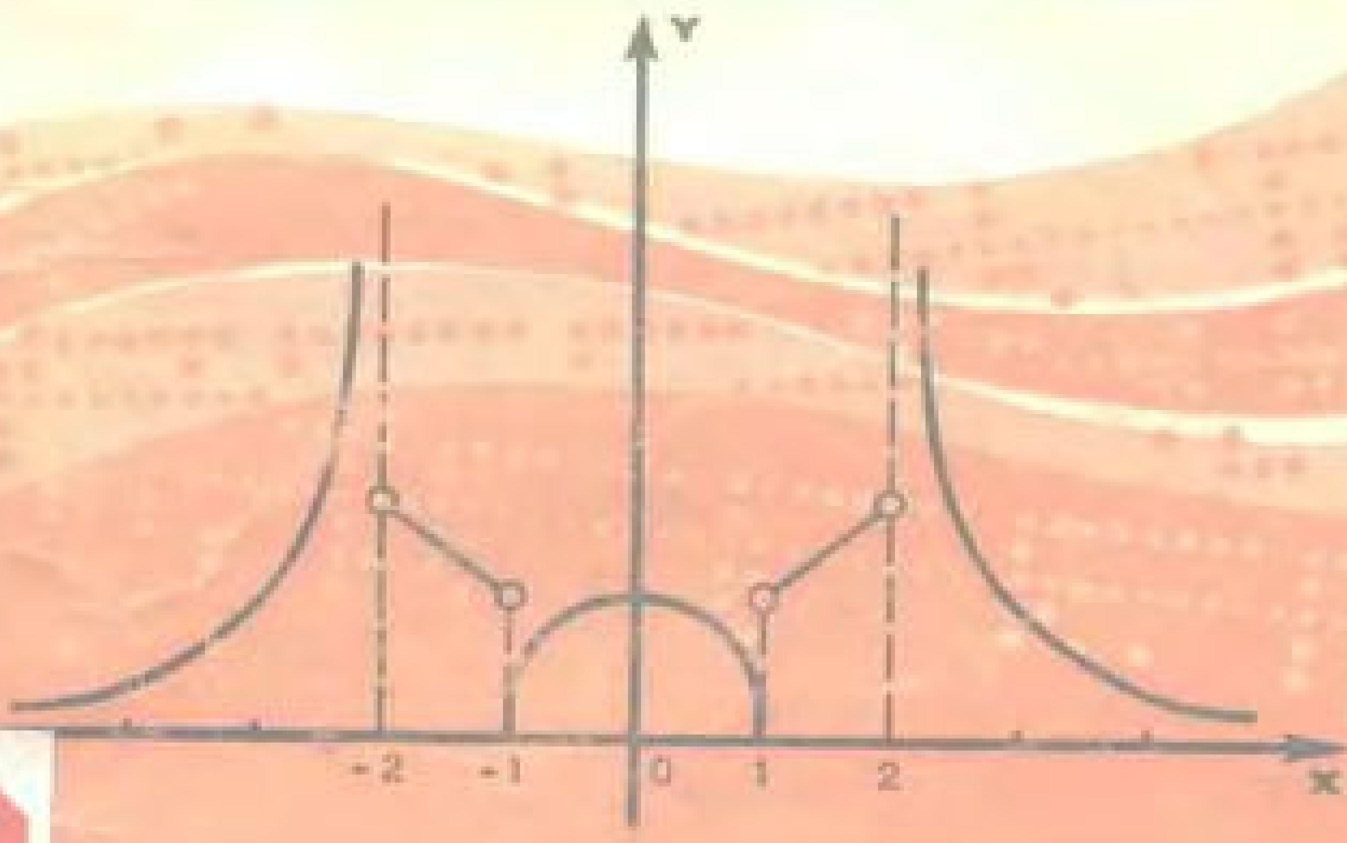
专业英语文选

数学专业英语文选

上册



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南京大学外文系公共英语教研室编

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前　言

为了响应党中央、华主席的伟大号召，实现我国四个现代化，特编辑本书，供有关同志提高阅读数学专业英语书刊的能力。

本书可供具有一定英语水平的数学专业学生和数学工作者阅读。

全书分上、下两册，每册各30课。内容由浅入深，以利读者循序渐进地阅读。每课课文后附有词汇、词组、注释，书末附有译文，便于读者自学时参考。

本书材料选自近代英美原著。专业内容较为广泛，包括初等、中等、高等数学各个方面。语言现象较为丰富多样，包括数学专业英语的各种句型结构及常用词汇。

本书在编写过程中，承蒙南京大学数学系周伯壎教授等同志指导与帮助，谨在此表示衷心的感谢。

由于编者的专业知识和语言水平的限制，本书一定还存在有不少缺点和错误，请读者提出宝贵的意见，以便修改和提高。

CONTENTS

1. MATHEMATICS COMES FROM PRACTICE

Engels said, "Like all other science, mathematics arose out of the needs of men." From the very① beginning, mathematics was the direct or indirect attempt to satisfy a definite need in production.

In his social practice, man began to feel the need of counting things and calculating the volume of a container. From this early need came the concepts of number and shape.② Then, geometry developed out of problems of measuring land, and trigonometry came from problems of surveying. To make calculation simpler,③ man learned to use symbols too, and algebra came into being as a result.

In elementary mathematics, we deal with constants only. With the rapid development of industry in 17th century, calculating with④ constants could no longer satisfy the needs of production. Many new problems in production called for a solution. To solve these problems, man began studying variable quantities and motion. This leap from constants to variable quantities brought about a new branch of mathematics — calculus.

In a word, mathematics comes from man's social practice. In studying mathematics,⑤ we must combine theory with practice. We must make mathematics serve⑥ socialist revolution and socialist construction of our country.

词　汇

mathematics [mæθi'mætiks] *n.* 数学
Engels ['eŋgəls] 恩格斯（人名）
arise [ə'raiz] *v.i.* 兴起；出现 (arose [ə'rəuz], arisen [ə'rizn])
need [ni:d] *n.* & *v.t.* 需要
direct [di'rekt] *a.* 径直的，直接的
indirect [indi'rekt] *a.* 间接的
attempt [ə'tempt] *n.* 企图
satisfy ['sætisfai] *v.t.* 满足，使（人）满意
definite ['definit] *a.* 明确的，确定的
social ['səuʃəl] *a.* 社会的
feel [fi:l] *v.t.* 觉得，感到 (felt [felt], felt)
count [kaunt] *v.t.* 数，计算
calculate ['kælkjuleit] *v.t.* 计算
volume ['vɔlju(:)m] *n.* 体积；容量
container [kən'teinə] *n.* 容器
concept ['kɔnsept] *n.* 概念
number ['nʌmbə] *n.* 数；数字；号码
shape [ʃeip] *n.* 形状
geometry [dʒi'ɔmitri] *n.* 几何学
problem ['prɔbləm] *n.* 问题
land [lænd] *n.* 土地
trigonometry [trigə'nɔmitri] *n.* 三角法，三角学
survey [sə:'vei] *v.t.* 测量
calculation [ˌkælkju'leiʃən] *n.* 计算
symbol ['simbəl] *n.* 记号；符号
algebra ['ældʒibrə] *n.* 代数学
result [ri'zʌlt] *n.* 结果；*v.i.* 由……产生；来自
elementary [ˌeli'mentəri] *a.* 初等的；基本的
deal [di:l] *v.i.* 处理；分派 (dealt [delt], dealt)
constant ['kɔnstənt] *a.* 不变的；*n.* 常量
century ['sentʃuri] *n.* 世纪
solution [sə'lu:ʃən] *n.* 解答；解决
variable ['vɛəriəbl] *a.* 可变的；*n.* 变量
quantity ['kwɔntiti] *n.* 量，数量
motion ['məuʃən] *n.* 运动
leap [li:p] *n.* 跳跃，跃进
branch [bra:ntʃ] *n.* 部门；分支；分科
calculus ['kælkjuləs] *n.* 微积分（学）
combine [kəm'bain] *v.t.* 使结合；合并
theory ['θiəri] *n.* 理论

词　组

out of 从……当中
(to) come into being 形成；产生
as a result 结果
(to) deal with 谈论；涉及
no longer 不再
(to) call for 需要
(to) bring about 引起；发生
in a word 总而言之
(to) combine ... with 使……与……结合；化合

注　释

① From the very beginning, ….

very 在这里为形容词，与名词连用，强调名词。

此部分译为：从一开始，……

② From this early need came the concepts of number and shape.

本句为倒装句。不倒装时应为：

The concepts of number and shape came from this early need.

From this early need 倒装在句首，表示强调 this early need；并承接上句意思。

③ To make calculation simpler 是不定式短语，作目的状语。

make 作为"使得"来翻译时，除了有宾语外，还要有宾语补足语。

此部分译为：为了使计算更为简单。

④ With the rapid development of industry in 17th century, calculating with constants …

第一个 with 译为：随着…

第二个 with 译为：用…

⑤ In studying mathematics, …

译为：在研究数学时，…

in 表示过程。

⑥ We must … of our country.

本句中 serve 为不定式，作 make 的宾语补足语。

… make mathematics serve …，译为：使数学为……服务。

不定式前一般都有不定式符号 "to"，但在 let, make, see, hear, watch, notice, feel 等少数一些动词后，作宾语补足语时，要省去不定式符号 "to"。

2. LANGUAGE OF MATHEMATICS

The language of mathematics is a language of signs and symbols. It is the same throughout the world.

Some of the best known symbols of mathematics are the Arabic numerals 1,2,3,4,5,6,7,8,9,0, the signs of addition "+", subtraction "−", multiplication "×", division "÷", and equality "=".

Arabic numerals are used because of their great convenience. In the number 111, three 1's① are used, and each has a different meaning. The 1 on the extreme right stands for the number one, the one in the second column from the right stands for the number ten, and the 1 in the third column stands for the number one hundred.

If we want to write the symbol for three tens, we put a 3 into the second column from the right. But we will not recognize it as a second column; we should write something down in the first column. Thus, it becomes necessary to think of three tens plus no ones, and to introduce a symbol to represent the absence of ones.② We use the symbol 0 for this purpose and call it zero. Zero plus any number gives that number again, and zero times any number gives zero.

词 汇

sign [sain] *n.* 记号；符号

throughout [θru:'aut] *prep.* 全；到处

best [best] *ad.* (well 的最高级) 最

最；最好

known [nəun] *a.* 大家知道的；已知的

Arabic ['ærəbik] *a.* 阿拉伯的

numeral ['nju:mərəl] *n.* 数字

addition [ə'diʃən] *n.* 加，加法

subtraction [səb'trækʃən] *n.* 减，减法

multiplication [ˌmʌltipli'keiʃən] *n.* 乘，乘法

division [di'viʒən] *n.* 除，除法

equality [i(:)'kwɔliti] *n.* 等号

use [ju:z] *v.t.* 用

convenience [kən'vi:niəns] *n.* 方便

each [i:tʃ] *pro.* 各，每

meaning ['mi:niŋ] *n.* 意义

extreme [iks'tri:m] *a.* 极端

right [rait] *n.* 右面；*a.* 右面的

column ['kɔləm] *n.* 行

recognize ['rekəgnaiz] *v.t.* 认出

something ['sʌmθiŋ] *n.* 某物；某事

thus [ðʌs] *ad.* 于是，因此

plus [plʌs] *prep.* 加，加上

introduce [ˌintrə'dju:s] *v.t.* 采用；介绍

represent [ˌrepri'zent] *v.t.* 代表

absence ['æbsəns] *n.* 无，缺

zero ['ziərəu] *n.* 零

times [taimz] *prep.* 乘

词　组

because of 因为

(to) stand for 代表

(to) think of 想

注　释

① In the number III, three 1's are used, ...

1's 表示数字 1 的复数形式。

省略号 "'" 可以用来构成做名词用的某些字母、数字、甚至标点符号的复数。如：

He drops his h's. 他常吞掉 h 音。

Count to ten by 2's. 请用 2 做单位，数到十。

Don't use too many !'s. 不要用这么多的！（惊叹号）号。

② Thus, it ... of ones.

it 为先行代词，是本句的形式主语。

to think ..., and to introduce ... ones. 为两个并列的不定式短语，作为本句的真正主语。

to represent the absence of ones 为另一不定式短语，是 to introduce ... 的目的状语。

3. MEASUREMENT

In the development of the physical sciences, we observed a rapid increase in scientific achievements after man began basing his conclusions upon experimental facts instead of upon inference.① Experimentation, however,② shows a quantitative study of some aspect of nature, and the important part of such a study is the measurement of the things with which it deals.③ Measuring any quantity means comparing it with an accepted unit as a standard, and finding out how many times larger or smaller it is than the standard unit.④ The length of an object is measured by finding how many times longer it is than some standard unit of length. For example, if this book were taken as a standard, and laid end to end five times along a desk surface, we know that the desk is 5 book-lengths long.⑤ If this book is laid down end to end five times and it does not quite reach the other end of the desk, we say that its length is a little over 5 books. In scientific work this "little over" part is not accurate enough. To be more accurate, we must measure what fractional part of the book the desk exceeds 5 book-lengths.⑥ If we measure the desk to be 1/5 of a book longer than the 5 book-lengths, we say its length as $5\frac{1}{5}$ or 5.2 book-lengths.⑦ A more accurate measnrement could be made by subdividing the book into ten equal parts. We would measure the desk to be a little more than 5.2 books long. Again we would have to

measure the fractional part of the subdivision by which the desk is longer than 5.2 book-lengths. If we found the fractional part as 1/2 a subdivision, we would write down a length of 5.25 books. The last measurement is obviously far more accurate than those⑧ for the larger units. The greater the accuracy needed, the smaller the subdivision must be.⑨

The weight of an object is similarly determined by finding how much heavier it is than some accepted standard weight unit. For example, if a piece of copper is four times as heavy as a standard pound,⑩ its weight is 4 pounds. Also, the smaller the subdivisions we have for the standard weight, the more accurate the weighing can be made.

词　汇

measurement [ˈmeʒəmənt] *n.* 度量,测量
physical [ˈfizikəl] *a.* 物理学(上)的;有形的;身体的
observe [əbˈzəːv] *v.t.* 观察;注意到
rapid [ˈræpid] *a.* 快的,急的
increase [ˈinkriːs] *n.* 增进,增加
scientific [ˌsaiənˈtifik] *a.* 科学的
achievement [əˈtʃiːvmənt] *n.* 完成;成就
base [beis] *v.t.* 把基础放在;*n.* 底
upon [əˈpɔn] *prep.*=on
conclusion [kənkˈluːʒən] *n.* 结束;结论
experimental [eksˌperiˈmentl] *a.* 实验的;经验的
fact [fækt] *n.* 事实
instead [insˈted] *ad.* 代替
inference [ˈinfərəns] *n.* 推论,推理
experimentation [eksˌperimenˈteiʃən] *n.* 实验(工作);实验法
however [hauˈevə] *conj.* 但是,然而;*ad.* 无论……也……
quantitative [ˈkwɔntiteitiv] *a.* 量的,定量的
aspect [ˈæspekt] *n.* 方面;样子
nature [ˈneitʃə] *n.* 自然;本性
compare [kəmˈpɛə] *v.t.* 比较,对照
accept [əkˈsept] *v.t.* 接受
unit [ˈjuːnit] *n.* 单位
standard [ˈstændəd] *n.* 标准
length [leŋθ] *n.* 长度
example [igˈzɑːmpl] *n.* 例如
surface [ˈsəːfis] *n.* 面,表面

lay [lei] *v.t.* 放
(laid [leid], laid)
reach [riːtʃ] *v.t.* 达到
accurate ['ækjurit] *a.* 准确;精密
enough [i'nʌf] *a.* 足够的; *ad.* 足够
fractional ['frækʃənl] *a.* 分数的;小数的
exceed [ik'siːd] *v.t.* 超过,胜过
subdivide ['sʌbdi'vaid] *v.t.* 细分;再分
equal ['iːkwəl] *a.* 相等的; *v.t.* 等于
subdivision ['sʌbdiviʒən] *n.* 细分;再分
obviously ['ɔbviəsli] *ad.* 显然
accuracy ['ækjurəsi] *n.* 正确,精确
weight [weit] *n.* 重量
object ['ɔbdʒikt] *n.* 事物;对象
similarly ['similəli] *ad.* 同样地;相似地
determine [di'təːmin] *v.t.* 决定;决心
heavy ['hevi] *a.* 重的,繁重的
piece [piːs] *n.* 块;片;件
copper ['kɔpə] *n.* 铜
pound [paund] *n.* 磅

词　　组

(to) base ... upon (on) 把基础放在……上;以……为根据
instead of 代替……;而不是……
(to) compare ... with 比较,对照
(to) find out 找出;求出
for example 例如
end to end 末端对末端,一端一端地
a little 少许,一点
far +比较级 ……得多

注　　释

① ... after man began ... instead of upon inference.

本句中 after 为连接词,连接一个时间状语从句。

两个 upon 并列。

② Experimentation, however, shows ...

本句 however 为连接词,译为然而……

它可位于句首、句中。如在句首,一般来说,后面有逗号“,”;如在句中,前后各用一逗号。

however 还可作副词,但其后面紧跟一形容词,此时译为:无论……。

如:…… however great ... 无论多么大;…… however small ... 无论多么小。

③ ... with which it deals.

为定语从句。修饰其前面的名词 things, deal with 连用。

④ Measuring any quantity ... than the standard unit.

本句为主从复合句。

自 Measuring ... 到 and finding out 为主句。

动名词短语 Measuring any quantity 为主语。

means 相当于联系动词。

comparing it ... and finding out 为两个并列的动名词短语，作表语。

as a standard 为介词短语，说明 an accepted unit.

how many times larger ... it is than ... unit 是由连接副词 how 引导的名词从句，作动名词 finding 的宾语。

此宾语从句中的主语、谓语动词 it is 倒装在表语…… larger or smaller 的后面。

than 为连接词，引导一省略比较状语从句。主语为 the standard unit。联系动词 is 和表语 large or small 省略。

⑤ For example, if this book were taken ..., and laid ... a desk surface, we know that ... long.

本句为主从复合句。

本句的特点是：自 if ... 至 surface 条件状语从句中，谓语动词为虚拟（假设）语气，表示与事实不符，或不大可能发生的事。

而自 we know 至句末中的谓语动词则为陈述语气。we know 为主语从句。

连接词 that 引导名词从句，作 know 的宾语。

as a standard 为介词短语，作主语补足语。

理科（科技）中这类词组很多。如：

to find ... as ... 求出……为……；发现……为……

to define ... as ... 把……定义为……

to consider ... as ... 把……认为是……

⑥ To be more accurate, ...

为不定式短语，作目的状语。放在句首，表示强调。

⑦ If we measure ... 5 book-lengths, ...

不定式短语 to be 1/5 ... 为宾语补足语。

⑧ those 为指示代词，代替前面提到的 measurement，以免重复。

⑨ The greater ..., the smaller ... must be.

本句为：The ＋形容词（或副词）比较级……（从句），＋the ＋形容词（或副词）比较级……（主句）的句型结构。

译为：愈（越）……愈（越）……。

先译从句，后译主句。

因为语气强，所以形成倒装句。

⑩ ... as heavy as ...

第一个 as 为副词，修饰作为表语的形容词 heavy。

第二个 as 为连接词，引导一省略比较状语从句。主语为 a standard pound。

联系动词 is 和表语 heavy 均省略。

as heavy as ... 译为如……一样重。

4. THE CIRCLE-MEASUREMENTS BY THE ANCIENT CHINESE MATHEMATICIANS

We already know that the ancient Chinese employed for π the value 3, or that they counted the circumference of a circle compared with diameter as 3 to 1.① The value of π was used in China as early at least as in the 12th century B.C. But the Chinese did not in any way remain satisfied with this rough value of π. Ever since then great efforts have been made to improve its accuracy and brilliant achievements obtained.②

Among the earliest Chinese circle-squarers mention must be made of Chang Hung in the first place. Chang was a famous scholar of the Han Dynasty. Chang's calculation of the circle, however, has been lost, although his value of π is given in a commentary on the "Arithmetic in Nine Sections" in the form that the ratio of the square of the circular circumference to that③ of the perimeter of the circumscribed square is 5 to 8. This is equivalent to taking π at $\sqrt{10}$.

In the period of the Three Kingdoms there lived another mathematician Liu Hui, in whose④ commentaries on the "Arithmetic in Nine Sections" we find the particulars of his quadrature of the circle.

Liu Hui starts, in his measurement of the circle, with a hexagon inscribed in a circle the diameter of which⑤ is taken as two feet. Each side of the hexagon is equal to

half the circular diameter. On this hexagon Liu Hui describes a dodecagon by doubling the number of its sides, and then doubles again the sides and describes a 24-gon, and so on. In this way the areas of the polygons thus formed gradually approach to the area of the circle, the difference diminishing step by step.⑥

Two centuries after Liu Hui, there appeared another and more distinguished circle-squarer, Tsu Chung-chih. Tsu contrived a more effective method of proceeding than his predecessors had followed, and obtained the accurate value for π. It was $\frac{355}{113}$. From this it is seen that China had possessed the accurate value for over 1,300 years before Europe, where⑦ the same value was obtained in 1855.

Tsu Chung-chih died in 500 at the advanced age of 71. His son Tsu Hong-chi was another distinguished mathematician following his father. It was he who⑧ first derived the world formula about the volume of a spherical ball, which is equal to $\frac{1}{6}\pi D^3$, where D denotes the diameter.

词　汇

circle [ˈsəːkl] *n.* 圆
ancient [ˈeinʃənt] *a.* 古代的
mathematician [ˌmæθiməˈtiʃən] *n.* 数学家
employ [imˈplɔi] *v.t.* 使用
value [ˈvæljuː] *n.* 值
circumference [səˈkʌmfərəns] *n.* 圆周
diameter [daiˈæmitə] *n.* 直径
early [ˈəːli] *ad.* & *a.* 早
least [liːst] *a.* 最小的；最少的
B.C. = **Before Christ** [biˈfɔː kraist] 公元前
remain [riˈmein] *v.i.* 仍然；剩下
rough [rʌf] *a.* 粗的；大约
ever [ˈevə] *ad.* 任何时候；始终
effort [ˈefət] *n.* 努力
improve [imˈpruːv] *v.t.* 改善，增进
brilliant [ˈbriljənt] *a.* 辉煌的
obtain [əbˈtein] *v.t.* 得到
among [əˈmʌŋ] *prep.* 在……之中
circle-squarer [ˈsəːkl ˈskweərə] *n.*

积圆家，求圆者
mention [ˈmenʃən] *n.* & *v.t.* 陈述，说到
Chang Hung 张衡（西汉学者，发明地球仪）
famous [ˈfeiməs] *a.* 有名的
scholar [ˈskɔlə] *n.* 学者
Han Dynasty [hæn ˈdainəsti] 汉朝
lose [luːz] *v.t.* 遗失
(lost [lɔst], lost)
commentary [ˈkɔməntəri] *n.* 注解
Arithmetic in Nine Section [əˈriθmetik in nain ˈsekʃən] 九章算术（中国古代算术书名）
ratio [ˈreiʃiou] *n.* 比，比例，比率
square [skwɛə] *n.* 正方形，平方；*v.t.* 自乘
circular [ˈsəːkjulə] *a.* 圆的
perimeter [pəˈrimitə] *n.* 周长
circumscribe [ˈsəːkʌmskraib] *v.t.* 在四周划线；使外切
equivalent [iˈkwivələnt] *a.* 相等于……的
period [ˈpiəriəd] *n.* 时期
Three Kingdoms [θriː ˈkiŋdəmz] 三国（指汉末蜀魏吴鼎立时期）
Liu Hui 刘徽（三国魏人）
particulars [pəˈtikjuləz] *n.* 详细情节（常用复数）
quadrature [ˈkwɔdrətʃə] *n.* 积圆法
hexagon [ˈheksəgən] *n.* 六边形
inscribe [insˈkraib] *v.t.* 使内接
foot [fut] *n.* 脚，足；英尺 (*pl.* **feet** [fiːt])
describe [disˈkraib] *v.t.* 记述，描述
dodecagon [dəuˈdekəgən] *n.* 十二边形
double [ˈdʌbl] *v.t.* 加倍
24-gon=polygon of 24 sides 二十四边形
polygon [ˈpɔligən] *n.* 多边形
gradually [ˈgrædjuəli] *ad.* 渐渐地
approach [əˈprəutʃ] *v.t.* & *v.i.* 接近；*n.* 入口；近路
difference [ˈdifərəns] *n.* 差别；差
diminish [diˈminiʃ] *v.t.* 减少
step [step] *n.* 步；台阶
appear [əˈpiə] *v.i.* 出现
distinguished [disˈtiŋgwiʃt] *a.* 有名的；卓越的
Tsu Chung-chih 祖冲之（南北朝时刘宋人，中国伟大科学家）
contrive [kənˈtraiv] *v.t.* 发明；设计
effective [iˈfektiv] *a.* 有效的
proceeding [prəˈsiːdiŋ] *n.* 进行；行动
predecessor [ˈpriːdisesə] *n.* 先辈
possess [pəˈzes] *v.t.* 具有，占有
Europe [ˈjuərəp] *n.* 欧洲
die [dai] *v.i.* 死
advanced [ədˈvaːnst] *a.* （年纪）老的；前进的
age [eidʒ] *n.* 年龄
Tsu Hong-chih 祖暅（读绠）之（发现球体体积公式）
derive [diˈraiv] *v.t.* 推导
formula [ˈfɔːmjulə] *n.* 公式
spherical [ˈsferikəl] *a.* 球的；天体的
denote [diˈnəut] *v.t.* 指示

词　　组

as early as 早在……时

at least 至少

any way 无论如何

ever since then 从……时候起

(to) make mention of 提到

in the first place 首先

and so on 依此类推

step by step 一步步地

注　　释

① 本句中两个连接词 that 引导两个并列的名词从句,作 know 的宾语。

as 3 to 1 为 counted 的宾语补足语。

分词短语 compared with diameter 作定语,修饰 circumference。

② 本为并列复合句，后一分句省略了和前一分句相同的助动词 have been，补全了应为 and brilliant achievements have been obtained。

③ 本句中第一个 that 为连接词,引导 the form 的同位语从句,至句末为止。

第二个 that 为指示代词,代替前面讲过的 the square, ,以避免重复。

④ 本句中 ... in whose ... of the circle 为定语从句,说明 Liu Hui。

whose 为关系代词的所有格,在从句中作定语。

⑤ 本句中 the diameter of which is taken as two feet 为定语从句,修饰 a circle。which 为关系代词,代表 a circle, of which 为介词短语,作定语,修饰 the diameter。

⑥ 本句中 the difference diminishing step by step 为独立(主格)分词结构,作状语。表示伴随动作或结果。

分词短语也可作状语用,但它的主语,即句子的主语。

而独立（主格）分词结构中则另有一主格（逻辑主语），不同于句子的主语。

这种结构有时还可作表示条件、时间等状语用。如:

Circumstance permitting, we shall begin to work tomorrow.

如情况允许,我们明天就开始工作。(表示条件)

The assembling of the machine completed, I started operating it.

机器安装完毕,我就开始操作。(表示时间)

⑦ where 为关系副词,引导定语从句,修饰 Europe。

⑧ It was he who first ... 为一种强调句型。此处强调句子的主语 he。

可译为:就是他(正是他)第一个……。

5. WHY DO WE COUNT THINGS IN GROUPS OF TEN

Why do we count things in groups of ten? The reason is that we have ten fingers. Long ago, when people had to count many things, they matched them against their fingers.① First they counted out enough things to match the fingers of both hands. Then they put these things aside in one group. If there were more than ten things to count,② they formed more groups. We might call our numbers two-handed numbers, because they grew out of counting things on two hands.

Some people had one-handed numbers, too. Because there are five fingers on one hand, they counted things out in groups of five.③ One-handed numbers were used by the people who lived in Italy over two thousand years ago. We call their written numbers Roman numerals, which we use even today. In Roman numerals, I stands for one, and V stands for five. To write six, the Romans wrote VI, which means five and one. A long time ago, when people did not wear shoes, they could use their toes for counting, too. So some people had a barefoot arithmetic. They counted things out in groups of twenty.

Sometimes people counted things in groups of twelve. We still use the twelve-in-one-group system for some purposes. When we measure time with a clock, we count the hours from one to twelve, and then start with one all

over again. The twelve-in-one-group system has left its mark on the English number-names, too. In English there is a word for each number from one to twelve. To name higher numbers above twelve, ten-in-one-group system is used.

Thousands of years ago there was a tribe who used to count things in groups of sixty. And we still use the sixty-in-one-group system when we measure time. There are sixty seconds in a minute, and there are sixty minutes in an hour.

词 汇

group [gruːp] *n.* 群，组
reason ['riːzn] *n.* 理由，原因
finger ['fiŋgə] *n.* 手指
ago [ə'gəu] *ad.* 以前
match [mætʃ] *v.t.* 相配，相称
aside [ə'said] *ad.* 旁边
might [mait] *v. aux.* (may 的过去时) 可以；*n.* 力
two-handed [tuː 'hændid] *a.* 两只手的
grow [grəu] *v.i.* 生长
(grew [gruː], grown [grəun])
Italy ['itəli] *n.* 意大利
Roman ['rəumən] *a.* 罗马的；*n.* 罗马人
even ['iːvən] *ad.* 甚至，也；*a.* 偶数的
wear [weə] *v.t.* 穿，戴
(wore [wɔː], worn [wɔːn])
shoe [ʃuː] *n.* 鞋
toe [təu] *n.* 脚趾
barefoot ['bɛəfut] *a.* 赤脚的
twelve-in-one *a.* 十二进位的，十二个一组的
system ['sistəm] *n.* 制度；体系
purpose ['pəːpəs] *n.* 目的；用途
measure ['meʒə] *v.t.* 计量；度量
mark [mɑːk] *n.* 记号，痕迹；*v.t.* 表示
ten-in-one *a.* 十进位的，十个一组的
tribe [traib] *n.* 部落
sixty-in-one *a.* 六十进位的，六十个一组的
second ['sekənd] *n.* 秒
minute ['minit] *n.* 分

词 组

long ago 老早(以前)
(to) have to 不得不，必须

(to) count out 点数

(to) put ... aside 把……放在一边

(to) grow out of 由……产生

all over again 从头到尾再来一遍

used to 习惯于

注 释

① to match ... against (或 with)把……与……相比。

against their fingers 为介词短语,作状语,修饰 match。

② to count 为不定式短语,作定语,修饰 things。

③ ..., they counted things out in groups of five.

……,他们用五个一组来数东西。

count out 数出。in 在此处译为"用"或"以"。in groups of five 用五个一组。此介词短语,作状语,修饰 counted。

6. COUNTING AND MEASURING IN OLD TIMES

If we could go back thousands of years, we would find that the people who lived then[1] had to think all the time of the necessary supplies of food and clothing, and of suitable shelter. So even primitive men were forced to think of questions such as the following: How many people must be fed? How much food and clothing do we need for the season? How long will our supplies last?

It is easy to see that, to answer these questions, they had to count and to measure.

In the early days people could not count as we do.[2] At first all counting was done with small stones, or sticks, or shells, or other convenient things. In the course of time, however, people learned to use their fingers in counting; and, since[3] we have ten fingers, the number ten became the foundation of all counting, in all parts of the world. This decimal system of counting led by slow degrees to our present method of reading and writing numbers and to the branch of mathematics which is now called arithmetic.

Nor could the primitive people measure as we do.[4] They knew nothing of rulers and yardsticks. Instead, they paced off distances or lengths or they used the hand for smaller measurements. As soon as farming became customary, more accurate methods of measurement arose. Naturally, the people wanted to know how large their farms

were, how much food they could grow and how much they could store in their barns, and how much water their wells would hold. These and many similar problems were gradually solved by the discovery of simple, practical rules. All this knowledge prepared the way for the branch of mathematics now called geometry.⑤

词　汇

back [bæk] *ad.* 向后，回头
supply [sə'plai] *n.* 供给；供应品
food [fu:d] *n.* 食品；粮食
clothing ['kləuðiŋ] *n.* （集合语）衣服
suitable ['sju:təbl] *a.* 适宜的
shelter ['ʃeltə] *n.* 窝棚；遮蔽物
primitive ['primitiv] *a.* 原始的
force [fɔ:s] *v.t.* 强迫
feed [fi:d] *v.t.* 喂养
(fed [fed], fed)
season ['si:zn] *n.* 季节
last [la:st] *v.t.* & *v.i.* 继续；持久
stone [stəun] *n.* 石头
stick [stik] *n.* 棍
shell [ʃel] *n.* 贝壳
convenient [kən'vi:njənt] *a.* 方便的
course [kɔ:s] *n.* 过程；方向
foundation [faun'deiʃən] *n.* 基础
decimal system of counting ['desiməl 'sistim əf 'kauntiŋ] 十进位制
slow [sləu] *a.* 慢的
degree [di'gri:] *n.* 次，次数；度（数）
present ['preznt] *a.* 现在的；今天的
yardstick ['ja:dstik] *n.* 码尺
pace [peis] *v.t.* & *v.i.* 步测（距离）
off [ɔ:f] *ad.* （和动词同用）离开
distance ['distəns] *n.* 距离
farming ['fa:miŋ] *n.* 农业；耕作
customary ['kʌstəməri] *a.* 通常的，习惯的
naturally ['nætʃrəli] *ad.* 自然地
store [stɔ:] *v.t.* 贮藏
barn [ba:n] *n.* 谷仓
well [wel] *n.* 井
hold [həuld] *v.t.* 容纳；装；认为
(held [held], held)
similar ['similə] *a.* 同样的；类似的
discovery [dis'kʌvəri] *n.* 发现
practical ['præktikəl] *a.* 实际的
rule [ru:l] *n.* 法则，规则
knowledge ['nɔlidʒ] *n.* 知识
prepare [pri'pεə] *v.t.* 准备

词　组

all the time 一直
such as 如同
in the course of 在……期间
by slow degrees 一点一点地

注　释

① If we could go back ..., we would find ... shelter.

此句为虚拟语气，不是真实条件句，说明不可能办到的事或与事实不相符合。

动词用时态代替法，即：原来如系陈述语气，则该用现在时态，现在虚拟语气则用过去时态（could 为 can 的过去式，would 为 will 的过去式）。说明与现在事实不符或现在不可能办到的事。

That 以下为名词从句，作 find 的宾语。who lived then 为定语从句，修饰 people。

"then" 为副词，此处译为"那时"或"当时"，作状语，修饰 lived。

② 此处 as 为连接词，引导一方式状语从句，译为：如（同）……一样。

在这从句中，do 为代动词，代替前面主句所用的谓语动词 count，避免重复。这是 do 的一种重要用法。

③ 此处 since 为连接词，引导一原因状语从句，译为：

因为……（或：由于……）。

④ Nor could ... as we do.

nor 开头的句子，因语气强，句子常倒装。此处助动词 could 倒装在主语 the primitive people 前面。

本句与上段中第一句前后相呼应。

上段第一句为：In the early days people could not count as we do.

⑤ ... now called geometry 为过去分词短语，作定语，修饰 the branch of mathematics。

7. EQUATION

An equation is a statement of the equality between two equal numbers or number symbols.①

Thus $r(r-5)=r^2-5r$ and $x-3=5$ are equations.

Equations are of two kinds — identities and equations of condition.②

An arithmetic or an algebraic identity is an equation. In such an equation either the two members are alike, or become alike on the performance of the indicated operation.③

Thus $12-2=2+8$; $(m+n)(m-n)=m^2-n^2$ are identities.

An identity involving letters is true for any set of numerical values of the letters in it.④

Thus the identity $x(r+2)=xr+2x$ becomes $3(7+2)=$ $=21+6$ or $27=27$, when, for example, $x=3$, and $r=7$.

An equation which is true only for certain values of a letter in it, or for certain sets of related values of two or more of its letters, is an equation of condition, or simply an equation.

Thus $3x-5=7$ is true for $x=4$ only; and $2x-y=10$ is true for $x=6$ and $y=2$ and for many other pairs of values for x and y.

A root of an equation is any number or number symbol which satisfies the equation.

To obtain the root or roots of an equation is called

solving an equation.⑤

There are various kinds of equations. They are linear equations, quadratic equations, etc.

词　汇

equation [iˈkweiʃən] *n.* 方程式，方程

statement [ˈsteitmənt] *n.* 陈述，叙述

identity [aiˈdentiti] *n.* 恒等式；一致

equation of condition [iˈkweiʃən əf kənˈdiʃən] 条件方程

arithmetic [əˈriθmətik] *a.* 算术的；*n.* 算术

algebraic [ˌældʒiˈbreiik] *a.* 代数学(上)的

member [ˈmembə] *n.* (方程的)端边；(团体的)一员，成员

alike [əˈlaik] *a.* 相同；*ad.* 一样

performance [pəˈfɔːməns] *n.* 执行，实行

indicate [ˈindikeit] *v.t.* 表示，指出

operation [ˌɔpəˈreiʃən] *n.* 运算；动手术

involve [inˈvɔlv] *v.t.* 包含

letter [ˈletə] *n.* 字；信

true [truː] *a.* 成立的；真正的；正确的

set [set] *n.* 组，套；集(合)

numerical [njuːˈmerikəl] *a.* 数的

certain [ˈsəːtin] *a.* 某一定的；确实的

relate [riˈleit] *v.t.* 使联系；使有关系

simply [ˈsimpli] *ad.* 只，仅仅

pair [pɛə] *n.* 一对

root [ruːt] *n.* 根

various [ˈvɛəriəs] *a.* 各种不同的

linear equation [ˈliniə iˈkweiʃən] 线性(一次)方程

quadratic equation [kwəˈdrætik iˈkweiʃən] 二次方程

etc. = et cetera [itˈsetrə] *conj.* 等等

词　组

either ... or 或……或；不是……就是

注　释

① ... between ... symbols 为介词短语，作定语，修饰 equality.

② ... of two kinds 为介词短语、作表语。identities and equations of condition 是 two kinds 的同位语，具体说明哪两类。

③ either ... or ... 是并列连接词，连接二并列谓语动词 are 和 become。

become 和 are 一样，也是联系动词。On the performance ... operation 中的 on 在此处引导一表时间的介词短语，作状语，修饰 become alike，译为：在……时候。

④ An identity involving letters ... of the letters in it.

involving letters 为现在分词短语，修饰 an identity。in it 为介词短语，作定语，修饰 the letters。

⑤ To obtain the root or roots of an equation 为不定式短语，此处作主语。

solving an equation 为动名词短语，作主语补足语。

8. THE USE OF EQUATIONS

When any two groups of terms or polynomials are written so that① one group equals the other, the expression is called an equation.

Equations are of very great use. With an equation we can work with② an unknown quantity, that is, we can easily find a way to bring that unknown quantity into a position where we can discover it. Doing that is called solving the equation. Let us start with very simple equations and try to find the ways of working with them. We can use equations in many arithmetic problems. We may notice that almost every problem gives us one or more statements that something is equal to something; this gives us equations, with which we may work if we need to.③

To solve an equation means to find the value of the unknown term. To do this, we must, of course, change the terms about until the unknown term stands alone on one side of the equation, thus making it equal to something on the other side. ④ We then get the value of the unknown and the answer to the question. To solve the equation, therefore, meaus to move and change the terms about without making the equation untrue, until only the unknown quantity is left on one side, no matter which side.⑤

词　汇

term [tə:m] *n.* 项；术语； *v.t.* 把……叫做

polynomial [ˌpɔliˈnəumiəl] *n.* 多项式

expression [iksˈpreʃən] *n.* 式

unknown [ˈʌnˈnəun] *a.* 未知的； *n.* 未知数

easily [ˈi:zili] *ad.* 容易地

position [pəˈziʃən] *n.* 位置；状态

discover [disˈkʌvə] *v.t.* 看出；发现

try [trai] *v.t.* 试

notice [ˈnəutis] *v.t.* 注意到

almost [ˈɔ:lməust] *ad.* 差不多；几乎

alone [əˈləun] *a.* 唯一，单独； *ad.* 单独

move [mu:v] *v.t.* 移动

untrue [ˈʌnˈtru:] *a.* 假的，不真实

leave [li:v] *v.t.* 离开；留下 (left [left]、left)

matter [ˈmætə] *n.* 事情；物质

词　组

(to) bring ... into ... position 使……处于……位置

of course 当然

(to) move and change the terms about 移项

no matter which 不论哪一个

注　释

① so that 为复合连接词，引导一目的或结果状语从句。译为："以便"或"结果"。

② 本句中两个 "with"，前者译为"用"；后者译为"对于"。
　　"with" 一词多意，要看具体上下文而定。

③ 在 We may ... we need to 这一句中，第一个 that 为连接词，引导名词从句，作 notice 的宾语，第二个 that 也为连接词，但引导的是 statements 的同位语从句，进一步说明它。
　　with which ... need to 为关系代词前有介词的一种类型的定语从句。with which 在从句中作状语，修饰 may work。if we need to 后面省略 work (或: to do so)，它是状语从句，修饰 may work。

④ ..., thus making it ... other side 为现在分词短语，作结果状语。

⑤ ..., no matter which side 后面省略了 it is。

9. RADICALS

A radical is an indicated root of an algebraic or arithmetic expression.

Thus $\sqrt{4}$, $\sqrt{7}$, $\sqrt[3]{5}$, and $\sqrt{r^2-3r-1}$ are radicals.

The small figure like the 3 in $\sqrt[3]{5}$ is called the index of the radical.

The index determines the order of the radical and indicates the root to be extracted.[1]

The radicand is the number or expression under the radical sign. In $\sqrt{8}$ and $\sqrt[3]{5am}$ the respective radicands are 8 and 5 am.

Radical expressions may be written in either of two ways: with radical signs or with fractional exponents.

Thus $\sqrt{7}$ and $7^{\frac{1}{2}}$ have the same meaning and $\sqrt[3]{c^2}$ equals $c^{\frac{2}{3}}$, etc.

A rational number is a positive or a negative integer or any number which can be expressed as the quotient, or ratio, of two such integers.[2]

Thus 8, -4, $\frac{3}{5}$, or 6.713 are rational numbers.

Any real number which is not rational is irrational.

If a number under a radical sign is such that[3] the root indicated cannot be exactly obtained, the radical represents an irrational number.

For example, $\sqrt{5}$ and $\sqrt[3]{7}$ are irrational.

A repeating decimal, though endless,[4] is not an irra-

tional number, for⑤ any repeating decimal can be expressed as a common fraction, and is therefore rational.

An indicated square root of a negative number is called an imaginary number.

All the numbers of algebra then may be placed in one or the other of the classes: real numbers and imaginary numbers.

Real numbers, as we have seen,⑥ are of two kinds, rational numbers and irrational numbers.

A surd is an irrational number in which the radicand is rational.

词 汇

radical ['rædikəl] *n.* 根式,根号; *a.* 根的

figure ['figə] *n.* 数字;图形

index ['indeks] *n.* 指数 (*pl.* indexes; indices ['indisi:z])

order ['ɔ:də] *n.* 次,序;界

extract [iks'trækt] *v.t.* 开(开平方根)

radicand ['rædikənd] *n.* 被开方数

respective [ris'pektiv] *a.* 各自的

exponent [eks'pəunənt] *n.* 指数,幂数

rational ['ræʃnl] *a.* 有理的

positive ['pɔzətiv] *a.* 正的;阳性的

negative ['negətiv] *a.* 负的;阴性的

integer ['intidʒə] *n.* 整数

express [iks'pres, eks-] *v.t.* 表示,用符号表示

quotient ['kwəuʃənt] *n.* 商

real ['riəl] *a.* 实的,真实的

exactly [ig'zæktli] *ad.* 正确地;整整地

irrational [i'ræʃnl] *a.* 无理的,不尽的

repeating decimal [ri'pi:tiŋ'desiməl] 循环小数

endless ['endlis] *a.* 无穷的,无限的

common ['kɔmən] *a.* 普通的;公约的

fraction ['frækʃən] *n.* 分数

imaginary [i'mædʒinəri] *a.* 虚的; *n.* 虚数

place [pleis] *v.t.* 放; *n.* 地方

surd [sə:d] *n.* 不尽根数; *a.* 不尽根的

词 组

(to be) of 具有

注　释

① to be extracted 为不定式短语之一般被动形式，此处作定语，修饰其前面名词 the root.

② of two such integers 为介词短语，作定语，修饰前面名词 quotient 和 ratio 两者，故在此介词短语前面有一逗号，以示与两个名词有关。

③ ... is such that .... such = such a number, that 为连接词，引导一个结果状语从句，说明 such。

④ though endless though 为连接词，引导一省略让步状语从句。形容词 endless 为表语。所省掉的主语是 it，即指前面主句中的主语 a repeating decimal，省掉的谓语动词则相应为为 is。

⑤ for 为并列连接词，引导句子，解释原因。译为："因为"。

⑥ 在 as we have seen 中，as 为关系代词，引导一非限制性定语从句，修饰全句。as 在从句中作 have seen 的宾语。

10. IRRATIONAL EQUATIONS

An irrational equation in one unknown is an equation in which the unknown occurs under a radical, or is affected by a fractional exponent.

Thus $x^2-2\sqrt{x}+1=0$, and $x^{\frac{1}{4}}-x^{\frac{1}{2}}+1=0$, and $y-(2y)^{\frac{1}{2}}-4=0$ are irrational equations.

One difficulty involved in the solution of such equations arises from the fact that sometimes results are obtained which do not satisfy the given equation and hence are not roots of that equation. A result of this kind is called extraneous.

Examples

(a) Solve $\sqrt{x-3}-5=0$.

Solution: Transposing,

$$\sqrt{x-3}=5. \quad (1)$$

Squaring, $x-3=25$ (2)

Solving: $x=28$

Check: $\sqrt{28-3}-5=0$

$\sqrt{25}-5=0$

$5-5=0$,

which is true.

(b) Solve $-\sqrt{x-3}-5=0$.

$$-\sqrt{x-3}=5. \quad (1)$$

$x-3=25$. (2)

$x=28$

$-\sqrt{28-3}-5=0$

$-\sqrt{25}-5=0$

$-5-5=0$.

which is not true.

It appears from a study of these solutions that① statements (1) differ only in the signs preceding their left members. Consequently this distinction disappears after

squaring, and equations (2) are identical. Since the remainder of the work in both (a) and (b) consists in the solution of (2), the result obtained is really the root of this equation. Whether the root obtained satisfies both (a) and (b), or only one of them, can be determined only by substitution. In this case it appears that (a) is an equation and that (b) is not, but is merely a false statement in the form of an equation.

In any case, all of the roots of the original equation are sure to be among the results found, provided② no result should be called a root unless③ it satisfies the original equation. This means that all results must be checked.

In irrational equations, as in all the work up to the present, it is understood that④ unless a radical or an expression affected by a fractional exponent is preceded by the double sign$\pm$, it has only the one value, just like any other number symbol.

Thus $\sqrt{81}$ means $+9$, and not -9.

Also $9^{\frac{1}{2}}$ means $+3$, while $-9^{\frac{1}{2}}$ means $-\sqrt{9}$, or -3; and $x^{\frac{1}{2}}$ means $+\sqrt{x}$ and not $-\sqrt{x}$.

If this fact is kept in mind, it is clear from an inspection of (b), above, that it could have no root, since the sum of two negative numbers could not possibly be zero.

The method of solving equations in which an unknown occurs under a radical is stated in the rule.

Transpose the terms so that one radical expression (the least simple one if there are two or more) is the only term in one member of the equation.

Next raise both members of the resulting equation to the same power as the index of the radical.⑤

Combine like terms in each member, and, if radical expressions still remain, repeat the two preceding operations, then solve this equation.

Check. Substitute in the original equation the values found⑥ and reduce the resulting numeric equation to its simplest form by extracting roots, but not by raising both members of the equation to any power.

Finally, reject all extraneous roots.

词 汇

occur [əˈkəː] *v.i.* 出现，发生
affect [əˈfekt] *v.t.* 影响
difficulty [ˈdifikəlti] *n.* 困难
given [ˈgivn] *a.* 已知的，已定的；假定的
hence [hens] *ad.* 因此
extraneous [eksˈtreinjəs] *a.* 外来的
transpose [trænsˈpəuz] *v.t.* 移项
check [tʃek] *v.t.* 核对，验算
differ [ˈdifə] *v.i.* 不同
preceding [priˈsiːdiŋ] *a.* 在前头的，上述的
left [left] *a.* 左边的
consequently [ˈkɔnsikwəntli] *ad.* 所以，因而
distinction [disˈtiŋkʃən] *n.* 差别
disappear [ˌdisəˈpiə] *v.i.* 消失
remainder [riˈmeində] *n.* 剩余；余数
consist [kənˈsist] *v.i.* 在于
really [ˈriəli] *ad.* 真正地，实在地
substitution [ˌsʌbstiˈtjuːʃən] *n.* 代入(法)，代换
merely [ˈmiəli] *ad.* 只不过
false [fɔːls] *a.* 假的，不成立的
original [əˈridʒənl] *a.* 原来的
sure [ʃuə] *a.* 肯定的，确定无疑的
provided [prəˈvaidid] *conj.* 倘若
unless [ənˈles] *conj.* 除非
precede [priˈsiːd] *v.t.* & *v.i.* 放在……之前，在……之前加上
while [(h)wail] *conj.* 而；当……的时候；虽则
mind [maind] *n.* 心；精神
clear [kliə] *a.* 清楚的
inspection [insˈpekʃən] *n.* 检查
sum [sʌm] *n.* 和，总数
possibly [ˈpɔsəbli] *ad.* 可能，也许
method [ˈmeθəd] *n.* 方法
state [steit] *v.t.* 叙述，陈述
next [nekst] *ad.* 其次
raise [reiz] *v.t.* 使自乘
like terms [laik təːmz] 同类项

reduce [ri'dju:s] *v.t.* 化，约；还原

numeric [nju:'merik] *n.* 数字，不能通约的数

finally ['fainəli] *ad.* 最后

reject [ri'dʒekt] *v.t.* 弃去

词　组

(to) consist in 在于

whether ... or 究竟，是否

in any case 无论如何；在任何情况下

up to the present 直到现在为止

(to) keep in mind 牢记（在心头）

(to) raise ... to ... power 把……乘到……幂次

注　释

① that 为连接词，引导一主语从句。it 为先行代词，作形式主语。

② provided 为过去分词，引导一条件状语从句。

③ unless 为连接词，引导一否定条件状语从句。

④ that 为连接词，引导一主语从句。it 为先行代词，作形式主语。

⑤ ... as the index of the radical 中，as 为关系代词，引导一定语从句，修饰 the same power，与 the same 连用。译为：和……同样。

as 也常与 such 连用，such ... as ... 译为：象……那样的。

这种定语从句常常是省略的。如本句中 the index of the radical 为主语，省略了谓语动词 is。

⑥ substitute 为及物动词，其宾语为 the values found。found 为 find 的过去分词，此处作定语，修饰 the values。宾语 the values 原应紧跟在动词 substitute 后面，但因它后面有过去分词 found 修饰它，如果调至前面，则本句将写成 substitute the values found *in* the *original* equation ...，结果会使人误（会）认为 in the original equation 作状语，修饰 found，那就错了。

11. IMAGINARIES

As① we have progressed in the study of algebra, nearly every forward step has involved the use of a more complicated and refined type of number. In arithmatic the positive integer and the positive fraction were sufficient for all requirements. In the beginning of algebra the negative integer and fraction were introduced. The solution of quadratic equations such as, $x^2-2=0$, forced upon us the irrational number, and led to the study of methods of operating with radicals, and to the solution of radical equations. Each new kind of number has presented itself as the root of an equation that② we were supposed to solve, and with the introduction of each number the power and generality of our algebraic method was increased.

Up to the present the square root of a negative number has been avoided, or dismissed with the remark that it is an imaginary③. It is true that we cannot imagine any length that the number $\sqrt{-2}$ could measure. The positive numbers are all that we need for measurement. Similarly tbe person with very simple mathematical needs might remark that, since positive integers are adequate for counting, the fractions and irrational numbers are unnecessary④.

Just as we have defined and used the operations of addition, subtraction, multiplication, and division with negative numbers and with irrational numbers, so⑤ now

we will define the meaning of these operations on the so called imaginaries. The introduction of these numbers enables us to solve completely the quadratic equation for all cases. They are frequently used in many branches of applied mathematics, especially in the theory of electricity.

The equation $x^2+1=0$, or $x^2=-1$, states that x is a number whose square is -1. By defining a new number, $\sqrt{-1}$, as one whose square is -1, we obtain one root for the equation $x^2+1=0$.

Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously $\sqrt{-5}$ means something very different from $\sqrt{5}$, and $\sqrt{-n}$ from $\sqrt{n}$⑥.

The positive numbers are all multiples of the unit $+1$ and the negative numbers are all multiples of the unit -1. Similarly, pure imaginary numbers are real multiples of the imaginary unit $\sqrt{-1}$, as $2\sqrt{-1}, 5\sqrt{-1}$ and $6\sqrt{-1}$.

Furthermore, $\sqrt{-4}=\sqrt{4\cdot(-1)}=2\sqrt{-1}$; $\sqrt{-a^2}=\sqrt{a^2(-1)}=a\sqrt{-1}$; and $\sqrt{-5}=\sqrt{5}\cdot\sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is $3\sqrt{-1}=3i$.

If a real number be⑦ united to a pure imaginary by a plus sign or a minus sign, the expression thus obtained is called a complex number.

词　汇

progress [prə'gres] *v.i.* 前进，进行

complicated ['komplikeitid] *a.* 复杂的

refine [ri'fain] *v.t.* 精炼，改良

sufficient [sə'fiʃənt] *a.* 足够的

requirement [ri'kwaiəmənt] *n.* 需

要
beginning [bi'giniŋ] *n.* 开始
suppose [sə'pəuz] *v.t.* 假定，想象
introduction [ˌintrə'dʌkʃən] *n.* 导言；介绍
generality [ˌdʒenə'ræliti] *n.* 普遍性，概括性
avoid [ə'vɔid] *v.t.* 避免
dismiss [dis'mis] *v.t.* 除去
remark [ri'mɑːk] *n.* 意见； *v.t.* 注意；陈述
imagine [i'mædʒin] *v.t.* 想象，推测
person ['pəːsn] *n.* 人
adequate ['ædikwit] *a.* 足够的
unnecessary [ʌn'nesisəri] *a.* 不必要的，多余的
define [di'fain] *v.t.* 下定义
enable [i'neibl] *v.t.* 使能
completely [kəm'pliːtli] *ad.* 十分，全然
frequently ['friːkwəntli] *ad.* 经常地
applied mathematics [ə'plaid, ˌmæθi'mætiks] 应用数学
especially [is'peʃəli] *ad.* 特别
theory of electricity ['θiəri əf ˌilek'trisiti] 电学理论
general ['dʒenərəl] *a.* 一般的，通常的
multiple ['mʌltipl] *n.* 倍数
pure [pjuə] *a.* 纯粹的；清的
furthermore ['fəːðə'mɔː] *ad.* 而且，更进一步
plus sign [plʌs sain] 加号
minus sign ['mainəs sain] 减号
complex number ['kɔmpleks 'nʌmbə] 复数

词　组

so called 所谓

in general 一般，一般说来

注　释

① As we ... of algebra, 中 as 为连接词，引导一时间状语从句。译为："当……时"。

② ... that we were supposed to solve 中，that 为关系代词，引导一定语从句，在从句中作 solve 的宾语。

③ ... with the remark 为介词短语，作状语，修饰谓语 has been avoided, or dismissed。 that 为连接词，引导 the remark 的同位语从句。

④ that 为连接词，引导宾语从句至句末为止，作 remark 的宾语。在宾语从句中又有两个句子。since 为连接词，所引导的句子到 for counting 为止，作为其后面句子的原因状语从句。

⑤ Just as ... irrational numbers 为比较状语从句。
just as 常与主句中 so 相配合使用，意为：正如……那样。

⑥ 形容词 different 是放在 something 这个代词后面来修饰它的。其它如 anything, everything, somebody 等少数几个代词，修饰它们的形容词，一律放在它们的后面，不能放在它们的前面。

... and $\sqrt{-n}$ from $\sqrt{n}$ 为一省略句，补充起来应为

... and $\sqrt{-n}$ means something very different from $\sqrt{n}$。

⑦ If a real number be united 中，所以用 be 是因为其前面省略了情态助动词 should。这种虚拟语气，表示语气的缓和。

12. LOGARITHMS

Logarithms were invented to shorten the work of extended numerical computations which involve one or more of the operations of multiplication, division, involution, and evolution. Their use has decreased the labor of computing to such an extent that many calculations which would require hours without the use of logarithms can be performed with their aid in a small fraction of that time①.

If we write the equation

$$n = b^a, \tag{1}$$

we express therein the essential relation between a number, n, and its logarithms, a, for a given base, b. In the notation of logarithms this is written

$$\log_b n = a, \tag{2}$$

and it is read "the logarithm of n to the base b equals a". We can define verbally in one statement both logarithm and base as follows:

The logarithm of a given number is the power to which another number, called the base, must be raised in order to equal the given number.

It is important to realize that equations (1) and (2) are merely two different ways of expressing precisely the same relations, one the exponential way, the other the logarithmic②. Above all it is necessary to keep in mind the fact that a logarithm is an exponent.

Thus in $81 = 3^4$, the given number is 81, the base is 3, and the logarithm is 4; that is, $\log_3 81 = 4$.

The base of the common system of logarithm is 10. Hence a table of common logarithms is really a table of exponents of the number 10. Since the greater portion of these exponents are approximate values of irrational numbers, it follows that computations by means of logarithms give only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table. The common system is used in numerical work almost exclusively.

The only other system of logarithms used in computations is called the natural system. It has for its base the irrational number 2.7182^+, which is usually denoted by the letter *e* and is used mainly for theoretical purposes.

It can be proved that the laws given before, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

词 汇

logarithm ['lɔgəriθəm] *n.* 对数
invent [in'vent] *v.t.* 发明
shorten ['ʃɔ:tn] *v.t.* 缩短,减少
extended [iks'tendid] *a.* 繁长的
computation [ˌkɔmpju'teiʃən] *n.* 运算,计算
involution [ˌinvə'lu:ʃən] *n.* 乘方
evolution [ˌi:və'lu:ʃən] *n.* 开方
decrease [di'kri:s] *v.t.* 减少
labor = labour ['leibə] *n.*劳动,工作
compute [kəm'pju:t] *v.t.* 计算
extent [iks'tent] *n.* 程度
require [ri'kwaiə] *v.t.* 需要
perform [pə'fɔ:m] *v.t.* 执行;完成
aid [eid] *n.* & *v.t.* 帮助
therein [ðɛər'in] *ad.* 在其中;在那里
essential [i'senʃəl] *a.* 主要的;根本的
relation [ri'leiʃən] *n.* 关系
notation [nəu'teiʃən] *n.* 记数法;表示法

verbally ['vəːbəli] *ad.* 逐字；口头
realize ['riəlaiz] *v.t.* 体会；认识
precisely [pri'saisli] *ad.* 正，恰；完全
exponential [ˌekspəu'nenʃəl] *a.* 指数的
logarithmic [ˌlɔgə'riθmik] *a.* 对数的
common system of logarithm 常用对数系
table [teibl] *n.* 表；桌子
portion ['pɔːʃən] *n.* 一部分
approximate value [ə'prɔksimeit 'væljuː] 近似值
exist [ig'zist] *v.i.* 存在
desire [di'zaiə] *v.t.* 希望，要求
proper ['prɔpə] *a.* 适当的；正常；真
exclusively [iks'kluːsivli] *ad.* 单独地，唯一地
natural system 自然对数系
theoretical [θiə'retikəl] *a.* 理论的
govern ['gʌvən] *v.t.* 支配，统治
assume [ə'sjuːm] *v.t.* 假定；呈现

词 组

as follows 如下
above all 尤其是，最重要的是

注 释

① ... to such an extent that ... 中，that 为连接词，引导一结果状语从句，至句末为止。此从句中又含有一个主句和一个定语从句，其主句中的主语是 many calculations，谓语动词是 can be performed。主、谓语之间隔有关系代词 which 所引导的定语从句。Their use 即指上句所讲的对数的使用。would 表示虚拟语气，介词短语 without the use of logarithms 表示条件。that time 即指 hours，避免重复。

② ... one the exponential way, the other the logarithmic 为两个省略句，说明前面提到的 two different ways，表示同位（进一步解释）。在 one 和 the other 后都省略了联系动词 is。在 logarithmic 后还省略了名词 way。

13. GEOMETRY AND GEOMETRICAL TERMS

Geometry is a branch of mathematics. This branch of mathematics is not chiefly concerned with numbers, although it uses numbers. It is not chiefly concerned with equations, although it also uses them. It is chiefly concerned with the study of forms, such as triangles, parallelograms, and circles.

A solid has length, breadth and thickness. A surface has length and breadth, but no thickness. A line has length but neither breadth nor thickness. A line may be a straight line or a curve. A point has position but not size.

A ray is a line beginning at a certain point and extending indefinitely. Two rays proceeding from the same point form an angle. The two rays are thus the arms or sides of the angle. If the arms of an angle extend in opposite directions in a straight line, the angle is called a straight angle. Half of a straight angle is called a right angle. An acute angle is an angle less than a right angle. An obtuse angle is greater than a right angle. A reflex angle is one greater than a straight angle.

A rectilinear figure of three sides is called a triangle. An isosceles triangle is one with two of its sides equal. An equilateral triangle is one with all its sides equal. We call a triangle a right triangle when one angle is a right angle. We call a triangle an obtuse triangle when one

angle is an obtuse angle. We call a triangle an acute triangle when all its angles are acute angles. We call a triangle equiangular triangle when all its angles are equal. In a right triangle the side opposite the right angle is called the hypotenuse.

A closed curve lying in a plane with all its points equally distant from a fixed point in the plane is called a circle. That fixed point is called the center of the circle. A straight line through the center and terminated at each end by the circle is called a diameter①. Half of the diameter is a radius. The length of the closed curve is called the circumference.

词 汇

geometrical [dʒio'metrikəl] *a.* 几何学的
chiefly ['tʃi:fli] *ad.* 主要地，多半地
concern [kən'sə:n] *v.t.* 从事；关于
triangle ['traiæŋgl] *n.* 三角形
parallelogram [,pærə'leləgræm] *n.* 平行四边形
solid ['sɔlid] *n.* 固体，立体； *a.* 固体的，立体的
breadth [bredθ] *n.* 宽
thickness ['θiknis] *n.* 厚
straight line [streit lain] 直线
curve [kə:v] *n.* 曲线； *a.* 弯曲的
point [pɔint] *n.* 点
size [saiz] *n.* 大小，尺寸
ray [rei] *n.* 射线；光线
extend [iks'tend] *v.t.* 伸展，延长
indefinitely [in'definitli] *ad.* 无限地
proceed [prə'si:d] *v.i.* 发生；进行；开始
angle ['æŋgl] *n.* 角，角度
arm [ɑ:m] *n.* 臂
side [said] *n.* 边
opposite ['ɔpəzit] *a.* 对向的，相反的
straight angle 平角
right angle 直角
acute angle [ə'kju:t 'æŋgl] 锐角
obtuse angle [əb'tju:s'æŋgl] 钝角
reflex angle ['ri:fleks 'æŋgl] 优角
rectilinear [,rekti'liniə] *a.* 直的，直线的
isosceles [ai'sɔsili:z] *a.* (三角形)二等边的，等腰的
equilateral ['i:kwi'lætərəl] *a.* 等腰的
hypotenuse [hai'pɔtinju:z] *n.* 斜边
closed [kləuzd] *a.* 封闭的

lie [lai] *v.i.* 在(某处)
(lay [lei], lain [lein], lying ['laiiŋ])
plane [plein] *n.* 平面
distant ['distənt] *a.* 远的;隔离
fix [fiks] *v.t.* 固定
center ['sentə] *n.* 中心,圆心
terminate ['tə:mineit] *v.t.* 终止
radius ['reidiəs] *n.* 半径

词 组

less than 少于;小于

注 释

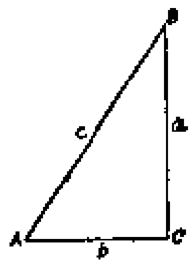
① through the center 为介词短语,作定语,修饰 line;
at each end 和 by the circle 为介词短语,作状语,修饰过去分词 terminated。

14. TRIGONOMETRIC FUNCTIONS AND SOLUTION OF RIGHT TRIANGLES

The sides and angles of a triangle are mutually dependent. We know this from geometry. Trigonometry begins by showing the exact nature of this dependence between the sides and angles of a triangle.① For this purpose trigonometry employs the ratios of the sides. These ratios are called trigonometric functions. The six trigonometric functions of any acute angle in a triangle, as A, are denoted as follows:

$\sin A$, read "sine of A"
$\cos A$, read "cosine of A"
$\tan A$, read "tangent of A"
$\csc A$, read "cosecant of A"
$\sec A$, read "secant of A"
$\cot A$, read "cotangent of A".

These trigonometric functions (ratios) are defined as follows (see figure):



(1) $\sin A = \dfrac{\text{opposite side}}{\text{hypotenuse}} \left(=\dfrac{a}{c}\right);$

(2) $\cos A = \dfrac{\text{adjacent side}}{\text{hypotenuse}} \left(=\dfrac{b}{c}\right);$

(3) $\tan A = \dfrac{\text{opposite side}}{\text{adjacent side}} \left(=\dfrac{a}{b}\right);$

(4) $\csc A = \dfrac{\text{hypotenuse}}{\text{opposite side}} \left(=\dfrac{c}{a}\right);$

(5) $\sec A = \dfrac{\text{hypotenuse}}{\text{adjacent side}} \left(= \dfrac{c}{b}\right);$

(6) $\cot A = \dfrac{\text{adjacent side}}{\text{opposite side}} \left(= \dfrac{b}{a}\right).$

These functions (ratios) are of fundamental importance in the study of trigonometry. They must be memorized.

One of the most important applications of trigonometry is the solution of triangles. Let us now take up the solution of right triangles. A triangle is composed of six parts, three sides and three angles. To solve a triangle is to find the parts not given. A triangle may be solved if three parts (at least one of these is a side) are given. A right triangle has one angle, the right angle, always given. Thus a right triangle can be solved when two sides, or one side and an acute angle, are given.

The general directions for solving right triangles are as follows.

(1) Draw a figure as accurately as possible representing② the triangle in question.

(2) When one acute angle is known, subtract it from 90° to get the other acute angle.

(3) To find an unknown part, select from (1) to (6) trigonometric functions (ratios), a formula involving the unknown part and two known parts, and solve for the unknown part.

(4) Check the values found, when they satisfy relations different from those used in the last step, they are correct. A convenient numerical check is the relation

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

词 汇

trigonometric [ˌtrigənəˈmetrik] *a.* 三角(法)的
function [ˈfʌŋkʃən] *n.* 函数
mutually [ˈmjuːtjuəli] *ad.* 相互地
dependent [diˈpendənt] *a.* 依赖,从属
exact [igˈzækt] *a.* 正确的
dependence [diˈpendəns] *n.* 依赖,从属
adjacent [əˈdʒeisənt] *a.* 邻接的
fundamental [ˌfʌndəˈmentl] *a.* 基本的
importance [imˈpɔːtəns] *n.* 重要,重要性
memorize [ˈmeməraiz] *v.t.* 记忆
application [ˌæpliˈkeiʃən] *n.* 应用
compose [kəmˈpəuz] *v.t.* & *v.i.* 组成
draw [drɔː] *v.t.* 画
(drew [druː], drawn [drɔːn])
accurately [ˈækjuritli] *ad.* 正确地;精密地
possible [ˈpɔsəbl] *a.* 可能
subtract [səbˈtrækt] *v.t.* 扣除,减去
select [siˈlekt] *v.t.* 选择
correct [kəˈrekt] *a.* 正确的

词 组

(to) take up 取,定
(to be) composed of 由……组成
in question 所讨论到的东西,所谈到的东西

注 释

① by showing the exact nature of ... 为介词短语,作状语,修饰 begins。showing 为动名词,作 by 的宾语,本身又带有宾语 the exact nature ...。
② as possible 为省略比较状语从句,修饰前一个 as。representing the triangle 为现在分词短语,作定语,修饰 figure。

15. GRAPHICAL REPRESENTATION OF TRIGONOMETRIC FUNCTIONS

A variable is a quantity to which an unlimited number of values can be assigned.① Variables are usually denoted by the later letters of the alphabet, as x, y, z.

A quantity whose value remains unchanged is called a constant. Numerical or absolute constants retain the same values in all problems, as 2, 5, $\sqrt{7}$, π, etc. Arbitrary constants are constants whose values are fixed in any particular problem. These are usually denoted by the earlier letters of the alphabet, as a, b, c, etc.

A function of a variable is a magnitude whose value depends on the value of the variable. Nearly all scientific problems deal with quantities and relations of this sort, and in the experiences of everyday life we are continually meeting conditions illustrating the dependence of one quantity on another. Thus, the weight a man is able to lift depends on his strength, other things being equal.② Hence we may consider the weight lifted as a function of the strength of the man. Similarly, the distance a boy can run may be considered as a function of the time. The area of a square is a function of the length of a side, and the volume of a sphere is a function of its diameter. Similarly, the trinomial

$$x^2 - 7x - 6$$

is a function of x because its value will depend on the

value we assume for x, and

$$\sin A, \quad \cos 2A, \quad \tan \frac{A}{2}$$

are functions of A.

The relation between the assumed values of a variable, and the corresponding values of a function depending on that variable, are very clearly shown by a geometrical representation where the assumed values of the variable are taken as the abscissas, and the corresponding values of the function as the ordinates of points in a plane. A smooth curve drawn through these points in order is called the graph of the function. Following are general directions for plotting the graph of a function.

First step. Place y equal to the function.

Second step. Assume different values for the variable $(=x)$ and calculate the corresponding result of the function $(=y)$, writing down the result in tabulated form.

Third step. Plot the points having the values of x as abscissas and the corresponding values of y as ordinates.

Fourth step. A smooth curve drawn through these points in order is called the graph of the function.

词　汇

graphical ['græfikəl] *a.* 图解的

representation [ˌreprizen'teiʃən] *n.* 图解；说明

unlimited [ʌn'limitid] *a.* 无限的；不定的；极大的

assign [ə'sain] *v.t.* 指定，确定

alphabet ['ælfəbit] *n.* 字母

absolute ['æbsəlu:t] *a.* 绝对的

retain [ri'tein] *v.t.* 保留

arbitrary ['a:bitrəri] *a.* 任意的

magnitude ['mægnitju:d] *n.* 量，数量，大小

depend [di'pend] *v.i.* 依赖，依靠

sort [sɔ:t] *n.* 种类

experience [iks'piəriəns] *n.* 经验，体验

continually [kən'tinjuəli] *ad.* 不断地，常常

illustrate ['iləstreit] *v.t.* 说明，举例证明；图解

able [eibl] *a.* 有才能的；能

lift [lift] *v.t.* 提起；抬起

strength [streŋθ] *n.* 力量，力气

area ['ɛəriə] *n.* 面积；区域

sphere [sfiə] *n.* 球体；区域

trinomial [trai'noumjəl] 三项式

corresponding [ˌkɔris'pɔndiŋ] *a.* 相应的

abscissa [æb'sisə] *n.* 横坐标 (*pl*, -sas; -sae [si:])

ordinate ['ɔ:dnit] *n.* 纵坐标

smooth [smu:ð] *a.* 平滑的

plot [plɔt] *v.t.* 绘制……的图；作(图)

graph ['grɑ:f] *n.* 图，图解

tabulate ['tæbjuleit] *v.t.* 列成表格

词　组

(to) depend on 依靠

(to be) able to 能……

in order 整整齐齐

注　释

① to which ... 至句末为定语从句，修饰 a quantity。此为关系代词前有介词的一种类型的定语从句。所用的介词往往和从句中的谓语动词有密切关系。此处用 to，即 assign ... to 的一种搭配使用。

② 本句为主、从复合句。Thus, the weight depends on his strength 为主句。a man is able to lift 为定语从句，修饰 the weight。其前面省去关系代词 that 或 which。

因为关系代词在从句中作 to lift 的宾语，所以可以省略，如果作主语，则不能省略。

other things being equal 为独立(主格)分词结构，作状语。

16. NUMBER AND NUMERALS

Let us first turn our attention to fractions. You have surely met the expressions "half" and "quarter". They are used when the denominator of the fraction equals 2 or 4. $\frac{1}{3}$ is read "one third". Other fractions are read in the same way. Thus we read $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{10}$, $\frac{1}{25}$, $\frac{1}{100}$ as one fifth, one sixth, one seventh, one tenth, one twenty-fifth and one hundredth. These expressions are regarded as nouns and may therefore① have a plural. Thus we read $\frac{2}{3}$ as two thirds; similarly $\frac{5}{6}$, $\frac{9}{10}$, $\frac{5}{100}$ are read as five sixths, nine tenths and five hundredths. However, if the last digit of the denominator is a 1 or a 2, then we do not read the fraction in the above-mentioned way. For example, we pronounce $\frac{5}{21}$ as "five over twenty-one". This method is also used in other cases. If the fraction is not a common one (e.g., $\frac{1}{1089}$ or $\frac{501}{1205}$), then we say "one over a thousand and eighty-nine" or "five hundred and one over twelve hundred and five".

Next, let us examine decimal fractions. They are very simple to pronounce.② You just read the integral part of the number in the ordinary way, then say "point" (stands for "decimal point") and then read the decimal places one

after the other. Thus 12.65 is read twelve-point-six-five. π correct to 6 decimal places, equals three-point-one-four-one-five-nine-two. Correct to five significant figures, it equals three-point-one-four-one-six. When the decimal fraction is smaller than one, it is not usual in England to write, for instance, 0.56, but only .56. .56 is read "point-five-six". .0007 is read "point-nought-nought-nought-seven" or more usually "point-three 0's-seven".

Now for algebraical expressions. Fractions are again read "over". $\frac{2a-1}{ax+b}$ is read $2a-1$ over $ax+b$ and brackets are indicated by the word "into", e.g. $(a+b)\ (a-b)$ is read "*a* plus *b* into *a* minus *b*". Powers are indicated by indices or exponents. The index 2 is read "squared" and the index 3 "cubed", or "to the third". Other indices are read "to the fourth, to the fifth, to the minus second, to the nth". The identity

$$a^3+b^3=(a+b)\ (a^2-ab+b^2)$$

reads "*a* cubed plus *b* cubed equals *a* plus *b* into *a* squared minus *ab* plus *b* squared". Or the equation

$$x^{-\frac{2}{3}}+\sqrt[5]{a^2}=0$$

reads "x to the minus two thirds plus the fifth root of *a* squared equals zero".

Some integral numbers are divisible only by themselves or by one, and they are called prime numbers. Some integral numbers cannot be expressed as a fraction of two prime numbers, and they are called irrational.

词　汇

first [fəːst] *ad.* 首先； *num.* 第一
turn [təːn] *v.t.* 转向
attention [əˈtenʃən] *n.* 注意(力)
surely [ˈʃuəli] *ad.* 必定，无疑
half [hɑːf] *n.* 一半，二分之一
quarter [ˈkwɔːtə] *n.* 四分之一；一刻钟
denominator [diˈnɔmineitə] *n.* 分母
one third [wʌn θəːd] 三分之一
regard (as) [riˈgɑːd] *v.t.* 把……看作……
noun [naun] *n.* 名词
plural [ˈpluərəl] *n.* 复数； *a.* 复数的
digit [ˈdidʒit] *n.* 数字
above-mentioned [əˈbʌv ˈmenʃənd] *a.* 上述的
pronounce [prəˈnauns] *v.t.* 发音
e.g. [ˈiːˈdʒiː] 例如
examine [igˈzæmin] *v.t.* 考察，检查
just [dʒʌst] *ad.* 只；正，刚
integral [ˈintigrəl] *a.* 整数的
ordinary [ˈɔːdnri] *a.* 普通的，平常的
significant [sigˈnifikənt] *a.* 有效的；有意义的；重要的
usual [ˈjuːʒuəl] *a.* 普通的，常见的
England [ˈiŋglənd] *n.* 英国
instance [ˈinstəns] *n.* 例
nought [nɔːt] *n.* 零
algebraical [ˌældʒiˈbreiikəl] *a.* 代数的
bracket [ˈbrækit] *n.* 括弧
into [ˈintu] *prep.* 向内；到……里
squared [skwɛəd] *a.* 平方的
cubed [kjuːbd] *a.* 立方的，三乘的
divisible [diˈvizibl] *a.* 除得尽的
prime number [praim ˈnʌmbə] 素数

词　组

one after the other 一个接一个地
correct to 正确到
for instance 例如

注　释

① therefore 可放在句首，也可放在句中，但一般仍先译。
② to pronounce 为不定式短语，作状语，修饰 very simple。

17. FRACTIONAL NUMBERS (I)

If a, b are two integers, a new number (a, b), or in ordinary notation $\frac{a}{b}$, is formed by the association of a and b, the new number being defined to be such as to satisfy the following conditions:①

(1) (a, b) is regarded as ordinally greater, equal to, or less than 1.② The expressions greater, equal to, or less than, are here used, not in their primitive sense as referring to maguitude, but in the sense in which we have used them in the case of integers, as assigning relative order to the numbers.

(2) $(a, 1)$ is defined as equal to a; thus if $b=1$, the association is regarded as equivalent to the integer a. Taking (1) in conjunction with this postulate, the new numbers have their orders assigned,③ not only relatively to one another, but relatively also to the integral numbers; so that the whole aggregate and fractions is ordered, in the sense that, of two given numbers, it can always be said which has the higber rank.

(3) The addition of two fractional numbers is defined by

$$(a, b)+(c, d)=(ad+bc, bd).$$

(4) The multiplication of fractional numbers is defined

$$(a, b)\times(c, d)=(ac, bd).$$

(5) The use of a fraction as an index is defined by

the postulate

$$x^{(a,b)} \times x^{(c,d)} = x^{(a,b)+(c,d)},$$

where x is any number, either integral or fractional. The symbol $x^{(a,b)}$ is to be interpreted subject to⑤ this postulate, in case such interpretation is possible.

It will be observed that, in the case $b=1$, $d=1$, the above definitions are consistent with those which have been adopted in the case of integral numbers; and thus the new numbers, together with the integers, form an aggregate with uniform laws of operations. It is easily seen that the operations with the new numbers satisfy the commutative, associative, and distributive laws. The inverse operation of division is now one which is always possible within the domain of the numbers; thus

$$(a,b)\div(c,d)=(ad,bc).$$

The inverse operation of subtraction, $(a,b)-(c,d)=(ad-bc,bd)$, is only possible if $(a,b)>(c,d)$.

The association of a pair of integers is a "number" in quite a different sense from that in which cardinal and ordinal numbers, hitherto discussed, are numbers. The justification of the extension of the term "number" to the fractions lies in the fact that a consistent scheme of operations can be imposed upon them of which the laws are in agreement with those which hold for operations which involve integers only.⑥

词 汇

association [əˌsəusi ˈeiʃən] *n.* 结合，联合

ordinally [ˈɔːdinli] *ad.* 顺次地

sense [sens] *n.* 意义；感觉

relative [ˈrelətiv] *a.* 有关系的；相对的

conjunction [kənˈdʒʌŋkʃən] *n.* 结合，连接
postulate [ˈpɔstjuleit] *n.* 公理，公设；*v.t.* 假定；要求
aggregate [ˈægrigeit] *v.t.* & *n.* 集合
ordered [ˈɔːdəd] *a.* 有序的
rank [ræŋk] *n.* 秩，次序；阶级
interpret [inˈtəːprit] *v.t.* 翻译；解释
subject [ˈsʌbdʒikt] *a.* 服从于；*n.* 主题；科目
interpretation [inˌtəːpriˈteiʃən] *n.* 翻译；解释
definition [ˌdefiˈniʃən] *n.* 定义
consistent [kənˈsistənt] *a.* 一致的
adopt [əˈdɔpt] *v.t.* 采用，采纳
uniform [ˈjuːnifɔːm] *a.* 一样的
law [lɔː] *n.* 法则；定律
commutative [kəˈmjuːtətiv] *a.* 交换的
associative [əˈsouʃiətiv] *a.* 结合的
distributive [disˈtribjuːtiv] *a.* 分配的
inverse [ˈinˈvəːs] *a.* 逆的
within [wiðˈin] *prep.* 在……面里
domain [dəuˈmein] *n.* 范围，领域
cardinal [ˈkaːdinl] *a.* 基数的；*n.* 基数
ordinal [ˈɔːdinl] *a.* 序数的
hitherto [ˈhiðəˈtuː] *ad.* 迄今
discuss [disˈkʌs] *v.t.* 讨论
justification [ˌdʒʌstifiˈkeiʃən] *n.* 认为正当；辩护
extension [iksˈtenʃən] *n.* 扩张，扩大
scheme [skiːm] *n.* 格式；计划
impose [imˈpouz] *v.t.* 强派
agreement [əˈgriːmənt] *n.* 一致；协约

词组

(to) refer to 指；涉及
in conjunction with 与……共同；与……连接着
subject to 服从于
(to) impose upon (on) 把……强加在……之上
in agreement with 符合；照
(to) hold for 适用于

注释

① the new number being defined to be such as to satisfy the following conditions 为独立(主格)分词结构。being defined 为被动语态的现在分词，to be such 为主语补足语。as to satisfy the following conditions 为结果状语，修饰 such。

② as ... greater, equal to, or less than 1 为三个并列的主语补足语。

③ have their orders assigned 是 "have + 宾语 + 过去分词(作宾语补足语)"的句型，表示动作要由别人而不是由自己来完成。如：

You shoud have the radio repaired.

你应该(请人)把无线电修好。

④ ... in the sense that, of two given numbers, it ... the higher rank 为由连接词 that 所引导的 sense 的同位语从句。of two given numbers 为定语,修饰 which。为了表示强调,所以倒装在前面。

⑤ subject to 为主语补足语。

⑥ 本句为主、从复合句。

主句: 自 The justification ... 到 in the fact。

从句: 连接词 that 引导 the fact 的同位语从句。在此从句中, 又包括三个从句:

第一个定语从句是: of which ... those; 第二个定语从句是: which hold for operations; 第三个定语从句是: which *involve* integers only.

18. FRACTIONAL NUMBERS (II)

The scheme which has been above indicated suffices for a formal definition and logical development of the properties of fractions, but it is subject to the objection that it is of an arbitrary character; indeed it is not easy to see why the particular laws of operation have been postulated, except as suggested by the traditional non-arithmetical conception of a fraction.[1]

To remedy this defect, a view of the nature of a fraction will be here given which relates the fraction with the process of counting, in such a manner that fractional and integral numbers have similar relations to that process. It will appear that the laws of combination given above naturally follow from this mode of regarding the fraction, with the exception of (5), which is however immediately suggested by the rule for integral indices.

Consider an aggregate of b objects, and out of these b objects pick out any a ($\leqq b$) of them. If we regard these a objects not only as single objects of number a, but also as belonging to an aggregate whose number a is associated with the cardinal number b of the aggregate to which they belong. This process being independent of the particular aggregate used,[2] the abstract fraction (a,b) is related to this process in an analogous manner to that in which the number b is related to the process of counting an aggregate whose cardinal number is b. Thus the fraction

(a, b), or $\frac{a}{b}$ is characteristic of an aggregate of a objects each of which belongs to an aggregate of b objects. The extension of the definition to the case $a>b$ is clear when we observe that it is unessential that the a objects taken should all belong to one and the same aggregate of b objects; it is sufficient that each of them be regarded as essentially belonging to some aggregate of cardinal number b. In accordance with this view, a fraction, say $\frac{3}{5}$,③ is characteristic of any three things each of which belongs to an aggregate of five things, i.e. $\frac{3}{5}$ means 3 out of 5. That the three things taken out of five should necessarily all be equal in respect of size, or some other kind of magnitude, is as irrelevant to the true nature of a fraction as the assumption of five things necessarily meaning five equal things is to the true nature of the number five.

词　汇

suffice [sə'fais] *v.t.* 足以，足够
formal ['fɔːməl] *a.* 正式的；外形的
logical ['lɔdʒikəl] *a.* 逻辑(上)的
property ['prɔpəti] *n.* 性质
objection [əb'dʒekʃən] *n.* 反对意见，异议
indeed [in'diːd] *ad.* 实在
except [ik'sept] *prep.* 除……以外；*conj.* 除非
suggest [sə'dʒest] *v.t.* 建议；暗示
traditional [trə'diʃənl] *a.* 传统的
non-arithmetical ['nɔnə'riθmetikəl] *a.* 非算术的
conception [kən'sepʃən] *n.* 概念
remedy ['remidi] *v.t.* 弥补；医治
defect [di'fekt] *n.* 缺点
view [vjuː] *n.* 观点；考察
process ['prəuses] *n.* 过程；方法
manner ['mænə] *n.* 做法；样子
combination [ˌkɔmbi'neiʃən] *n.* 结合，组合
mode [məud] *n.* 方式，方法
exception [ik'sepʃən] *n.* 例外
immediately [i'miːdjətli] *ad.* 立即
pick [pik] *v.t.* 挑
single ['siŋgl] *a.* 单个的；纯粹的；

v.t. 挑选

belong [bi'lɔŋ] *v.i.* 属于

independent [ˌindi'pendənt] *a.* 与……无关

abstract ['æbstrækt] *a.* 抽象的; *n.* 抽象观念

analogous [ə'næləgəs] *a.* 与……相似,类似……的

characteristic [ˌkæriktə'ristik] *a.* 特有的; *n.* 特征,特色

unessential ['ʌni'senʃəl] *a.* 非本质的;非必要的

essentially [i'senʃəli] *ad.* 本质地;必不可少地

accordance [ə'kɔːdəns] *n.* 一致,调和

necessarily ['nesisərili] *ad.* 必然;必要

respect [ris'pekt] *n.* 关系;尊敬

irrelevant [i'relivənt] *a.* 没关系的

assumption [ə'sʌmpʃən] *n.* 假设

词 组

(to) follow from 作为当然的结果而成

(to) pick out 挑选

(to) belong to 属于

(to be) independent of 与……无关

(to be) analogous to 与……相似,类似

in accordance with 依照,与……一致

in respect of 关于

(to be) irrelevant to 与……不相干,与……没关系

注 释

① except as suggested ... a fraction, 这里 as suggested ... a fraction 是介词 except 的宾语。可以说 as suggested = that they have been suggested, 这样, as suggested ... a fraction 就是一个名词从句了。其实, except 后面的宾语可以有很多方式,除一般名词、代词以外,可以用介词短语,连接副(代)词加不定词构成的短语,还可以用从句,等。如:

I shall not go except together with him.

He told us nothing except how to go there.

② This process ... used 为独立(主格)分词结构,作原因状语。

这种结构可放在句首或句末,但须用逗号和句子的其它部分分开。

③ say $\frac{3}{5}$ 为插入语,译为: 例如 $\frac{3}{5}$ (或译为: 譬如说 $\frac{3}{5}$)。

19. THE METHOD OF LIMITS

The method of limits, which is essential both to pure Analysis and to the applications of Analysis in Geometry and in Kinetics, had a geometrical origin in the Method of Exhaustions, which was applied by the Greek geometers to determine lengths, areas, and volumes, in simple cases. This method, supplemented by the notion of the numerically infinite, was developed in later times, in various forms, into a general method which formed the basis of the Infinitesimal Calculus. The traditional geometrical conception of a limit may be exemplified by the case of the determination of the length of a curve as the limit of a sequence of properly chosen inscribed polygons. The lengths of the perimeters of the polygons are regarded as continually approaching the required length of the curve, whilst① the number of sides of the polygons is continually increased, and the maximum length of the sides of a polygon is diminished indefinitely. The limit, the length of the curve, is then regarded as actually reached at the end of a process described as making the number of sides of the polygon infinite; this mode of attainment of the limit being however inaccessible to the sensuous imagination, and disguising an actual qualitative change of a geometrical figure, which possesses corners and is bounded by segments of straight lines, into② one which has no corners and has a curvilinear boundary. No doubt was felt as to

the existence of the limit, which was regarded as obvious from geometrical intuition. That a curve possesses a length of an area was considered to require no proof. The first mathematician who recognized the necessity for a proof of the existence of a limit was Cauchy, who gave a proof of the existence of the integral of a continuous function. That the logical basis of the traditional method of limits is defective has in recent times received a posteriori confirmation by the exhibition of continuous function which possesses no differential coefficient, and by many other cases of exception to what were regarded as ordinary results of analysis resting on the method of limits, which have been brought to light by those mathematicians who have been engaged in examining the foundations of analysis.③

The arithmetical theory of limits, which is summed up in the general principle of convergence, provides a definite criterion for the existence of the limit of a sequence of numbers; and a considerable part of modern analysis is concerned with obtaining special forms of the general criterion adapted upon the theory of irrational numbers; for, in default of an arithmetical theory of irrational numbers, all attempts to prove the existence of a limit of a convergent sequence are doomed to inevitable failure; and this for the simple reasou that a convergent sequence of rational numbers does not necessarily possess④ a limit which is within the domain of such numbers. The definition of real numbers by means of convergent sequences of rational numbers is not a mere postulation of the existence of limits to such sequences; it involves rather the introduction of an enlarged conception of number, of such a character that

the scheme of ordered real numbers should form a consistent whole, and such that every convergent sequence of numbers in the domain of real number necessarily has a limit within that domain. The postulation of the existence of the aggregate of real numbers is justified by showing that a complete scheme of definitions and postulates can be set up for the elements of this aggregate, and that such a scheme does not lead to contradiction. As regards the existence of limits in the case of lengths, areas, volumes, and etc., referred to above, the order of procedure is a reversal of the traditional one, the existence of the limit being no longer inferred from geometrical intuition. For example, in the case of the determination of the length it is not assumed to be independently known to exist, but is defined as the arithmetical limit of the sequence of numbers which represent the perimeters of a suitable sequence of inscribed polygons. When this sequence is convergent, and its limit is independent of the particular choice of the polygons, subject to a suitable restriction, then the limit so obtained determines the length of the curve. In case⑤ no such limit exists, the curve is regarded as not having a length.

词　汇

limit ['limit] *n.* 极限
analysis [ə'næləsis] *n.* 分析；解析
kinetics [kai'netiks] *n.* 动力学
origin ['ɔridʒin] *n.* 起点；原点
the method of exhaustion [ðə'meθəd əf ig'zɔːstʃən] 穷举法
apply [ə'plai] *v.t.* 应用
Greek [griːk] *a.* 希腊(人)的
geometer [dʒi'ɔmitə] *n.* 几何学家
supplement ['sʌplimənt] *v.t.* 补足，补充
numerically [njuː'merikəli] *ad.* 用数；数字上
infinite ['infinit] *n.* 无穷(大)；无

穷(多)

infinitesimal calculus [infini'tesimәl 'kælkjulәs] 微积分学

exemplify [ig'zemplifai] *v.t.* 例证,例解

determination [di,tә:mi'neiʃәn] *n.* 决定

sequence ['si:kwәns] *n.* 序列,次序

properly ['prɔpәli] *ad.* 适当地

inscribed polygon 内接多边形

whilst [hwailst] *conj.* = while 当……的时候;而

maximum ['mæksimәm] *a.* 最大的;最多的

attainment [ә'teinmәnt] *n.* 成就

inaccessible ['inæksesәbl] *a.* 难达到的

sensuous ['sensjuәs] *a.* 感觉的

imagination [i,mædʒi'neiʃәn] *n.* 假想;想象

disguise [dis'gaiz] *v.t.* 装作;隐藏(真相等)

actual ['æktjuәl] *a.* 实际的

qualitative ['kwɔliteitiv] *a.* 性质上的;定性的

corner ['kɔ:nә] *n.* 隅角

bound [baund] *v.t.* 以……为界;邻接

segment ['segmәnt] *n.* (线)分;弓形;圆缺

curvilinear [kә:vi'liniә] *a.* 曲线的

boundary ['baundәri] *n.* 分界;界线

doubt [daut] *n.* 疑问

existence [ig'zistәns] *n.* 存在

obvious ['ɔbviәs] *a.* 明显的;明白的

intuition [,intju'iʃәn] *n.* 直觉;直观

proof [pru:f] *n.* 证明;试验

Cauchy 柯西(人名)

continuous [kәn'tinjuәs] *a.* 连续的

defective [di'fektiv] *a.* 有缺陷的

receive [ri'si:v] *v.t.* 得到

a posteriori ['eipɔs,teri'ɔ:rai] *a.* 后验的,经验的,归纳的

confirmation [,kɔnfә'meiʃәn] *n.* 确定;证实

exhibition [eksi'biʃәn] *n.* 提出;显示;展览

differential [,difә'renʃәl] *a.* 微分的; *n.* 微分

coefficient [,kәui'fiʃәnt] *n.* 系数

rest [rest] *v.i.* 靠,躺

engage [in'geidʒ] *v.t.* 使从事

arithmetical [,æriθ'metikәl] *a.* 算术的

principle ['prinsәpl] *n.* 原理;本质

convergence [kәn'vә:dʒәns] *n.* 收敛

provide [prә'vaid] *v.t.* & *v.i.* 预备;供应

criterion [krai'tiәriәn] *n.* 标准 (*pl.* criteria [krai'tiәriә])

considerable [kәn'sidәrәbl] *a.* 重要的;不少的

special ['speʃәl] *a.* 特别的,专门的

adapt [ә'dæpt] *v.t.* 适应;修改

default [di'fɔ:lt] *n.* 缺乏

prove [pru:v] *v.t.* 证明

convergent [kәn'vә:dʒәnt] *a.* 收敛的

doom [du:m] *v.t.* 命定

inevitable [in'evitәbl] *a.* 不可避免

的

failure [ˈfeiljə] *n.* 失败

mere [miə] *a.* 单单的

postulation [ˌpɔstjuˈleiʃən] *n.* 假定；要求

enlarge [inˈlɑːdʒ] *v.t.* 扩大

character [ˈkæriktə] *n.* 性质，特征

whole [həul] *n.* 整体

justify [ˈdʒʌstifai] *v.t.* 证明；以为正当

complete [kəmˈpliːt] *a.* 完全的；圆满的

element [ˈelimənt] *n.* 要素；原理；初步

contradiction [ˌkɔntrəˈdikʃən] *n.* 矛盾

procedure [prəˈsiːdʒə] *n.* 程序

reversal [riˈvəːsəl] *n.* 颠倒，反转

infer [inˈfəː] *v.t.* & *v.i.* 推理；表示

independently [ˌindiˈpendəntli] *ad.* 独立；任意

choice [tʃɔis] *n.* 选择

restriction [risˈtrikʃən] *n.* 限制，约束

词 组

(to) bring to light 暴露、公开，发现

(to be) engaged in 从事于，正做着

(to) sum up 总结

in default of 因无……；若缺少……时

(to be) doomed to failure 注定要失败

by means of 借助于……

in the case of 至于；在……情况下

in case 要是，一旦

注 释

① whilst ... indefinitly 中，whilst (=while) 为并列连接词，引导的句子与上句并列，表示同时存在的两种事物的对比（或和伴随发生的动作）。如：

Some experiments are difficult while (whilst) others are easy.

有些实验是难的，而其它实验是容易的。

② into one 为 change 所要求的另一个介词短语，change of ... into ...。

③ 本句为主、从复合句，共有五个从句。

That ... is defective 为连接词 that 所引导的主语从句，作整个复合句的主语。

谓语动词为 has received。

which ... coefficient 为关系代词 which 引导的定语从句。

what ... limits 为连接代词 what 引导介词 to 的宾语从句。

which ... mathematicians 为关系代词 which 引导的定语从句。

who ... analysis 为关系代词 who 引导的定语从句。

④ 连接词 that 引导 reason 的同位语从句。

⑤ in case 为连接词，引导一条件状语从句。

20. LIMITS AND INFINITY

A number which can take on successively different numerical values is called a variable. It should be remembered that, although it is common for the successive values of the variable to be related to each other in accordance with some law, these values need not have any definite relations one to another.① As stated before a variable is commonly represented by a letter such as x, y, or z.

Thus in the equation

$$ax^2+bx+c=0,$$

the variable is x. In this case the successive values of the variable are related to each other by the law expressed in the algebraic equation.

The numerical value of the recurring decimal .9999 ... is a variable depending on the number of 9's added to the series on the right. Every 9 thus added increases V, and the number of 9's which may be so added is unlimited. In spite of this, V always remains less than 1, although approaching more and more closely to that value. Here the number 1 is called the limit of the variable V.

If a variable V takes on successively a series of values which approach closer and closer to a fixed number L in such a manner that the numerical value of $V - L$ becomes and remains less than any finite number however small, then V is said to approach the limit L②.

This may be written limit of $V = L$.

The symbol $\rightarrow$ gives us the equivalent notation $V \rightarrow L$, which is read V approaches L as a limit.

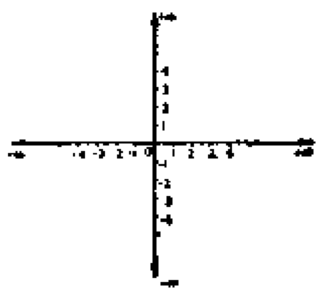
If a variable n takes on in succession all the values 1,2,3,4,..., we can conceive of no final value for n, since the system of natural numbers is unlimited. Here we may say that *n increases without limit* or *n becomes infinite*. This is stated in the following.

Definition. If a variable n becomes and remains greater than any positive number K, however great, we say n increases without limit, or n becomes infinite.

The usual symbol for a variable which has become infinite is the sign ∞, read infinity.

Infinity is not a number in the sense in which $3, \sqrt{2}$ and -9 are numbers. It is greater than any number. For present purposes it must be regarded as a figure of speech rather than as a number that can be added, subtracted, multiplied, or divided③. In fact, the symbol ∞ cannot be operated upon according to the laws governing the fundamental operations④.

We may have a negative infinity as well as positive one. In order to indicate the range of values which x and y may take in graphical work, the axes are often marked as in the following figure.



A constant number, however large, is never spoken of as infinite.

If the variable n in $\frac{1}{n}$ takes on in succession the values 1,2,3,4,..., no final value of $\frac{1}{n}$ can be imagined.

But as the variable n increases without limit, $\frac{1}{n}$ becomes very small and approaches nearer and nearer to zero without ever actually becoming zero.

In general, if a in the fraction $\frac{a}{n}$ is any constant not zero, and n a variable increasing without limit, $\frac{a}{n}$ approaches zero as a limit.

词 汇

infinity [in'finiti] *n.* 无穷大
successively [sək'sesivli] *ad.* 连续地
remember [ri'membə] *v.t.* 记住
commonly ['kɔmənli] *ad.* 通常，一般
recurring decimal [ri'kɔ:riŋ 'desiməl] 循环小数
add [æd] *v.t.* 加
series ['siəri:z] *n.* 级数 (*sing.* & *pl.*)
close [kləus] *ad.* 接近
finite ['fai'nait] *a.* 有限的
conceive [kən'si:v] *v.t.* 想象，想出
final ['fainl] *a.* 最后的
speech [spi:tʃ] *n.* 发言
rather ['rɑ:ðə] *ad.* 宁可
multiply ['mʌltiplai] *v.t.* 乘
divide [di'vaid] *v.t.* 除
operate ['ɔpəreit] *v.t.* & *v.i.* 起作用；运行
according [ə'kɔ:diŋ] *ad.* 依照
range [reindʒ] *n.* 范围；连续；限度
axis ['æksis] *n.* 轴 (*pl.* axes ['æksi:z])
near [niə] *ad.* 接近

词 组

one to another 互相
in spite of 尽管
in such a manner 以这样一种方式
in succession 接连
(to) conceive of 想出
in the sense 从……意义上讲
a figure of speech 一种比喻，一种说法
rather than 宁愿……不愿；而不
(to) operate upon 起作用，生影响
according to 依照
as well as 和

注　释

① It 为先行代词，指 that 所引导的主语从句、主语从句本身为主从复合句。连接词 although 引导的是让步状语从句，these values ... another 为主句。

在让步状语从句中，it 为先行代词，代表 to be related ... some law，作形式主语。

② 本句为主、从复合句。

If ... small 为条件状语从句。在此从句中，which ... manner 是定语从句，修饰 values; that ... small 为连接词 that 引导的结果状语从句，修饰 such; than ... small 是由连接词 than 引导的省略比较状语从句，修饰 less。

however small 修饰 any finite number。

③ rather than 为连接词，连接两个 as 所引导的介词短语，作为 it 的主语补足语。that 为关系代词，引导定语从句，修饰 a number。

④ 本句也可写成：In fact we cannot operate upon the symbol ∞ according to the laws governing the fundamental operations.

这样就可知道 upon 这个词是有宾语 the symbol 的，但因变成被动语态的句型之后，其宾语已变成主语了。

其他动词词组如 speak of, depend on (upon), look after 等都有这种情况。

21. FUNDAMENTAL IDEAS IN PLANE ANALYTIC GEOMETRY

Introduction

The development of analytic geometry is generally credited to two French mathematicians, Fermat (1601–1665) and Descartes (1596–1650). The main characteristic of analytic geometry is that it uses algebra in the study of geometry. A less emphasized but nonetheless important characteristic is that it permits geometrical visualization of algebraic relationships. Historically, the development of analytic geometry opened the door to the development of the differential and integral calculus, which many regard as the basic mathematical tool of the scientific revolution, which started in the seventeenth century.

Coordinates

The key idea in analytic geometry is the establishment of a one-to-one correspondence between the points in a plane and pairs of numbers (x,y). Choose two perpendicular lines and a unit of length.

Following convention, we call the horizontal axis the x-axis, the vertical axis the y-axis, and their point of intersection the origin.① It is also customary to regard the positive direction of the x-axis as being to the right, and the positive direction of the y-axis as being up. Each axis is scaled off in terms of the chosen unit length so that the

letter O is associated with the origin, and the symbol $+a$, is associated with a distance a units to the right of the origin on the x-axis, or a units above the origin on the y-axis. Similarly, $-a$ is associated with a distance a units to the left of the origin on the x-axis, or a units below the origin on the y-axis. In this way a one-to-one correspondence has been set up between all the real numbers and the points on the x-axis, and between all the real numbers and the points on the y-axis.

We will associate with any point in the plane a pair of numbers by the following rule: We draw perpendiculars from the point to each of the two axes. If these lines intersect the x-axis at a, and the y-axis at b, we assign the pair of numbers (a, b) to the point. We call (a, b) the coördinates of the point P and we designate this fact by writing $P(a, b)$. The symbol a attached to the x-axis is called the x-coordinate, or abscissa, of P; the number b attached to the y-axis is called the y-coordinate, or ordinate, of P. Conversely, if we were to locate the point corresponding to $P(a, b)$, we would draw a line parallel to the y-axis through the point marked a on the x-axis, and draw another line parallel to the x-axis through the point marked b on the y-axis.③ The point of intersection of these two parallel lines would be labeled $P(a, b)$.

The two axes divide the plane into four quadrants, called respectively the first, second, third, and fourth quadrants and labeled I, II, III, and IV. Points in the first quadrant have both coordinates positive; in the second quadrant the x-coordinate is negative and the y-coordinate is positive; in the third quadrant both coordinates are ne-

gative, and in the fourth quadrant the x-coordinate is positive and the y-coordinate is negative.

Sets of Points in the Plane

We have already shown that there is a one-to-one correspondence between points in a plane and pairs of numbers (x,y). Certain sets of points in the plane may be of special interest. For example, we may wish to examine the set of points comprising the circumference of a certain circle, or the set of points constituting the interior of a certain triangle. One may wonder if③ such sets of points may be succinctly described in a compact mathematical notation.

We may write

$$\{(x, y) \mid y = 2x\} \qquad (1)$$

to describe the set of ordered pairs (x, y), or corresponding points, such that the ordinate is equal to twice the abscissa. In effect, then, the vertical line in (1) is read "such that". By "the graph of the set of ordered pairs" is meant the set of all points of the plane corresponding to the set of ordered pairs. The student will readily infer that the set of points constituting the graph lies on a straight line.

Consider the set

$$\{(x, y) \mid y = x^2\}$$

consistent with our previous interpretation, this symbol represents the set of ordered pairs (x, y) such that the ordinate is equal to the square of the abscissa. Here, the total graph comprises a simple recognizable geometrical figure, a curve known as a parabola.

On the basis of these two examples, one may be tempted to believe that any arbitrarily drawn curve, which of course determines a set of points or ordered pairs, could be described succinctly by a simple equation. Unfortunately, this is not the case.

Consider now the set

$$\{(x, y) \mid y > 2x\} \tag{2}$$

to describe the set of points (x, y) whose ordinate is greater than twice its abscissa. In this case, our set of points constitutes not a curve, but a region of the coordinate plane.

词 汇

idea [aiˈdiə] *n.* 想法；概念

plane analytic geometry [plein ænəˈlitik dʒiˈɔmitri] 平面解析几何

credit [ˈkredit] *v.t.* 归功于人；*n.* 信任；荣誉

French [frentʃ] *a.* 法国的；*n.* 法国人

Fermat 费马（人名）

Descarts 笛卡儿（人名）

emphasize [ˈemfəsaiz] *v.t.* 强调，着重

nonetheless [nʌnðəˈles] *ad.* & *conj.* 虽然……但是；仍然

permit [pə(ː)ˈmit] *v.t.* 允许

visualization [ˌvizjuəlaiˈzeiʃən] *n.* 具体化

relationship [riˈleiʃənʃip] *n.* 关系

historically [hisˈtorikəli] *ad.* 在历史上；从历史观点上说

basic [ˈbeisik] *a.* 基本的

tool [tuːl] *n.* 工具

coordinate [kəuˈɔːdnit] *n.* 坐标；*a.* 坐标的

key [kiː] *n.* 关键；钥匙

establishment [isˈtæbliʃmənt] *n.* 建立

one-to-one *a.* 一对一的

correspondence [korisˈpondəns] *n.* 对应，一致

perpendicular [ˌpəːpənˈdikjulə] *a.* 垂直的；*n.* 垂线

convention [kənˈvenʃən] *n.* 惯例

horizontal [hɔriˈzɔntl] *a.* 水平的；*n.* 水平线

vertical [ˈvəːtikəl] *a.* 垂直的；*n.* 垂线

intersection [ˌintəˈsekʃən] *n.* 交叉

direction [diˈrekʃən] *n.* 方向

scale [skeil] *v.t.* 标度 *n.* 标度，比

例

associate [ə'səuʃieit] *v.t.* 使联合；连带

below [bi'ləu] *ad.* 在下；*prep.* 在……之下

intersect [ˌintə'sekt] *v.t.* 与……交叉

designate ['dezigneit] *v.t.* 指示；把……叫做

attach [ə'tætʃ] *v.i.* 附着，相连

conversely ['kɔnvəːsli] *ad.* 逆，相反地

locate [ləu'keit] *v.t.* 放；设计

parallel ['pærəlel] *a.* 平行的；(to) 平行于

label ['leibl] *v.t.* 附标记

quadrant ['kwɔdrənt] *n.* 象限

respectively [ris'pektivli] *ad.* 各自；分别

interest ['intrist] *n.* 利害；兴趣

comprise [kəm'praiz] *v.t.* 包含；由……组成

constitute ['kɔnstitjuːt] *v.t.* 构成

interior [in'tiəriə] *n.* 内部；*a.* 内部的

wonder ['wʌndə] *v.t.* & *v.i.* 惊奇；不知道

succinctly [sək'siŋktli] *ad.* 简明地

compact ['kɔmpækt] *a.* 简洁的；*n.* 条约

twice [twais] *ad.* & *n.* 两倍；再

effect [i'fekt] *n.* 效果；*v.t.* 实现，引起

readily ['redili] *ad.* 容易地；不勉强地

consider [kən'sidə] *v.t.* 考虑；料想

recognizable ['rekəgnaizəbl] *a.* 可认识的；可承认的

parabola [pə'ræbələ] *n.* 抛物线

basis ['beisis] *n.* 基础，根本 (*pl.* bases ['beisiːz])

tempt [tempt] *v.t.* 尝试

believe [bi'liːv] *v.t.* & *v.i.* 相信

unfortunately [ʌn'fɔːtʃnitli] *ad.* 不巧

region ['riːdʒən] *n.* 区域

词 组

(to) credit to 把(成功)归(某人)

(to) scale off 标度

in terms of 依……，用……的字眼

(to be) attached to 隶属于

in effect 事实上

on the basis of 以……为基础

This is not the case. 情况不是这样。

注 释

① the vertical axis the *y*-axis

即：We call the vertical axis the *y*-axis.

and their point of intersection the origin

即：and we call their point of intersection the origin.

② Conversely, if we were to locate ..., we would draw a line ...

本句为虚拟语气，用过去时态代替现在时态，说明与现在事实不符，或不可能成立的假设。

③ 此处 if = whether，译为：是否。

一般用在 ask, see, try, doubt, learn, wonder 等字之后。

22. CONIC SECTIONS

Introduction

A conic section is defined to be any of the curves of intersection of a plane with a cone. For convenience we shall use pairs of right circular cones.

If the plane of intersection is parallel to the base of the cone, the curve of intersection is a circle.

If the plane of intersection cuts one cone of the pair without intersecting the other, and if the plane is not parallel to the base, the curve is called an ellipse.

If the plane of intersection is parallel to an element of the cone, the curve of intersection is called a parabola.

If the plane of intersection is such that it intersects both cones, the pair of curves of intersection is called a hyperbola.

We leave it to the student to confirm that the intersection of the plane with the cone may in certain degenerate cases give a point, a line, or two lines.

These curves were investigated as early as 350 B.C. by the Greek mathematician Menaechmus, but it was Appollonius of Perga who, more than a hundred years later, analyzed them in thorough detail by means of the painstaking and laborious methods of Euclidean geometry. The work of Appollonius on conics earned for him the name of "Oreat Geometer." The methods employed by Appollonius are frighteningly disagreeable by modern standards.

The development of coordinate geometry by Fermat and Descartes has reduced this difficult problem of conic sections to one that is readily accessible to a moderately intelligent student.

The properties of the conics deduced by Appollonius represent his special perceptions and ingenuity. Coordinate geometry, however, permits the deduction of infinitely many properties of conics in addition to those of Appollonius. Many of the properties that he neglected have turned out to have far greater importance in applications than the ones he deduced.

The curves have been singled out for special study not only because of their historical importance but because there are a number of physical phenomena that are well described by these curves. For instance, the path a planet follows in its revolution about the sun is an ellipse. The path that a projectile follows when influenced only by *gravity* is parabolic. The paths of some comets are hyperbolic. In our problem sets, we shall encounter other physical situations in which the conic sections arise.

Since each conic section possesses infinitely many properties, any one of which① is sufficient to define it uniquely, we will choose as the defining property for each a property closely related to its practical use.

词 汇

conic section [ˈkɔnikˈsekʃən] 锥线，二次曲线

cone [kəun] *n.* 圆锥，锥形

right circular cone 直立圆锥

cut [kʌt] *v.t.* 切，割 (cut, cut)

ellipse [iˈlips] *n.* 椭圆，椭圆形

hyperbola [haiˈpəːbələ] *n.* 双曲线

confirm [kənˈfəːm] *v.t.* 证实；确定

degenerate [diˈdʒenərit] *a.* & *n.* 退

化(的);变了质(的)
investigate [in'vestigeit] *v.t.* 研究;调查
Menaechmus 米尼克穆斯(人名)
Appollonius 阿波罗纽斯(人名)
Perga 沛加(地名)
analyze ['ænəlaiz] *v.t.* 分析
thorough ['θʌrə] *a.* 彻底的;完全的
detail ['di:teil] *n.* 详细
painstaking ['peinzteikiŋ] *a.* 劳苦的
laborious [lə'bɔ:riəs] *a.* 辛苦的;困难的
Euclidean geometry ['ju:klidjən dʒi'ɔmitri] 欧几里得几何学
conic(s) *n.* 锥线法(论)
earn [ə:n] *v.t.* 博得(名声);挣得
frighteningly ['fraitniŋli] *ad.* 令人吃惊地
disagreeable [ˌdisə'griəbl] *a.* 难对付的;不愉快的
accessible [æk'sesibl] *a.* 进得去的,易受影响的
moderately ['mɔdəritli] *ad.* 适用;普通
intelligent [in'telidʒənt] *a.* 有智力的
deduce [di'dju:s] *v.t.* 推论;推断
perception [pə'sepʃən] *n.* 知觉;理解
ingenuity [ˌindʒi'nju:iti] *n.* 发明才能;技巧
deduction [di'dʌkʃən] *n.* 推理
infinitely ['infinitli] *ad.* 无限地
neglect [ni'glekt] *v.t.* 忽视
historical [his'tɔrikəl] *a.* 历史的,历史上的
phenomenon [fi'nɔminən] *n.* 现象(*pl.* phenomena)
path [pɑ:θ] *n.* 路
planet ['plænit] *n.* 行星
revolution [ˌrevə'lu:ʃən] *n.* 革命;公转
projectile ['prɔdʒiktail] *n.* 抛物体;子弹
influence ['influəns] *n.* 影响
gravity ['græviti] *n.* 重力,地心吸力
parabolic(al) [ˌpærə'bɔlik(əl)] *a.* 抛物线的
comet ['kɔmit] *n.* 彗星
hyperbolic(al) [ˌhaipə:'bɔlik(əl)] *a.* 双曲线的
encounter [in'kauntə] *v.t.* 遇见
situation [ˌsitju'eiʃən] *n.* 情况
uniquely [ju:'ni:kli] *ad.* 唯一地

词　组

in thorough detail 详细地
(to) turn out 结果是
(to) single out 挑选

注　释

① ..., any one of which ... 为一定语从句。
此处关系代词和 of 组成介词短语,在从句中,作定语,修饰 any one。

23. APPLICATIONS OF MATRICES

In recent years the applications of matrices in mathematics and in many diverse fields have increased with remarkable speed. Matrix theory plays a central role in modern physics in the study of quantum mechanics. Matrix methods are used to solve problems in applied differential equations, specifically, in the area of aerodynamics, stress and structure analysis. One of the most powerful mathematical methods for psychological studies is factor analysis, a subject that makes wide use of matrix methods. Recent developments in mathematical economics and in problems of business administration have led to extensive use of matrix methods. The biological sciences, and in particular genetics, use matrix techniques to good advantage. No matter what① the student's field of major interest is, a knowledge of the rudiments of matrices is likely to broaden the range of literature that he can read with understanding.

In this section we will give some elementary examples of how matrices are utilized.

The solution of n simultaneous linear equations in n unknowns is one of the important problems of applied mathematics. Descartes, the inventor of analytic geometry and one of the founders of modern algebraic notation, believed that all problems could ultimately be reduced to the solution of a set of simultaneous linear equations. Although

this belief is now known to be untenable, we know that a large group of significant applied problems from many different disciplines are reducible to such equations. Many of the applications, require the solution of a large number of simultaneous linear equations, sometimes in the hundreds. The advent of computers has made the matrix methods effective in the solution of these formidable problems.

Example 1. Solve the simultaneous equations for x_1, x_2 and x_3.

$$2x_1+3x_2+4x_3=4$$
$$2x_1+x_2+x_3=-2$$
$$-x_1+x_2+2x_3=2$$

Solution. We may rewrite these equations in matrix form

$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \tag{1}$$

and call the matrix of coefficients A, the 3×1 matrix of unknowns x, and the 3×1 matrix on the right k. We may then write Equation (1) in the form

$$Ax=k \tag{2}$$

If it were possible to find a 3×3 matrix, which is designated by A^{-1} and is called the inverse of matrix A, such that

$$A^{-1}A=I \tag{3}$$

where I is the identity matrix, then we would multiply both members of Equation (2) by A^{-1}. Equation (2) would then become

$$A^{-1}Ax=A^{-1}k \tag{4}$$

Using Equation (3), we could rewrite Equation (4) as

$$Ix = A^{-1}k$$

$$x = A^{-1}k \tag{5}$$

Specifically for this case, without telling you how we get it,

$$A^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{pmatrix}$$

Using this in Equation (5), we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)(4)+(2)(-2)+(-1)(2) \\ (5)(4)+(-8)(-2)+(-6)(2) \\ (-3)(4)+(5)(-2)+(4)(2) \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 24 \\ -14 \end{pmatrix}$$

Thus $x_1 = -10$, $x_2 = 24$, and $x_3 = -14$. From the above discussion, we see that the problem of solving n simultaneous linear equations in n unknowns is reduced to the problem of finding the inverse of the matrix of coefficients. It is therefore not surprising that in books on the theory of matrices the techniques of finding inverse matrices occupy considerable space. Of course, we will not in our limited treatment discuss such techniques. Not only are matrix methods useful in solving simultaneous equations, but they are also useful in discovering whether or not the set of equations are consistent, in the sense that they lead to

solutions, and in discovering whether or not the set of equations are determinate, in the sense that they lead to unique solutions.

Another area of application of matrices lies in transformation theory. The following are the transformation equations of rotation

$$x = x\cos\theta + y\sin\theta$$
$$y = -x\sin\theta + y\cos\theta$$

These equations may be written in matrix form as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In general, we can regard any 2×2 matrix, say

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

as transforming a point (x,y) into a point (X,Y) in accordance with the matrix equation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

A complicated transformation frequently may be regarded as a series of simple transformations. In such cases it is possible to represent this series of simple transformations by a single matrix. Consider a series of transformations accomplished by successive multiplications of the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by 2×2 matrices, A_1, A_2, and A_3. These successive multiplications will produce a point that we can denote by the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. We then denote the transformation by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A_3\left\{A_2\left[A_1\begin{pmatrix} x \\ y \end{pmatrix}\right]\right\} \tag{7}$$

Since matrix multiplication is associative, we may re-

write Equation (7) as

$$\begin{pmatrix}X\\Y\end{pmatrix}=(A_3A_2A_1)\begin{pmatrix}x\\y\end{pmatrix}$$

or

$$\begin{pmatrix}X\\Y\end{pmatrix}=A\begin{pmatrix}x\\y\end{pmatrix}$$

where A is the single matrix which is the product of the matrices A_3, A_2, A_1.

Example 2. A series of transformations is performed on the column vector $\begin{pmatrix}x\\y\end{pmatrix}$ corresponding to the matrices

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix},\ \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix},\ \text{and}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}.$$

Express the series of transformations by means of a single transformation matrix and show the graphic meaning of each of these transformations.

Solution.

$$\begin{aligned}\begin{pmatrix}X\\Y\end{pmatrix}&=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\\
&=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}(-1)(1)+(0)(0) & (-1)(0)+(0)(-1)\\(0)(1)+(1)(0) & (0)(0)+(1)(-1)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\\
&=\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\\
&=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\\
&=\begin{pmatrix}x\\y\end{pmatrix}\end{aligned}$$

Thus the series of transformations bring the point (x,y) back to itself.

词　汇

matrix ['meitriks] *n.* 矩阵 (*pl.* matrices ['meitrisi:z])
diverse [dai'və:s] *a.* 不同的
field [fi:ld] *n.* 范围；田地
remarkable [ri'mɑ:kəbl] *a.* 显著的；惊人的
speed [spi:d] *n.* 速度
central ['sentrəl] *a.* 中心的，重要的
role [rəul] *a.* 作用；角色
modern ['mɔdən] *a.* 近代的
quantum mechanics ['kwɔntəm mi'kæniks] 量子力学
specifically [spi'sifikəli] *ad.* 特别；明确地
aerodynamics [ˌεərəudai'næmiks] *n.* 空气动力学，气体力学
stress [stres] *n.* 应力
structure ['strʌktʃə] *n.* 结构
powerful ['pauəful] *a.* 有力的；有功效的
psychological [ˌsaikə'lɔdʒikəl] *a.* 心理学(上)的
factor analysis ['fæktə ə'næləsis] 因子分析，因素分析
wide [waid] *a.* 广阔的
economics [ˌi:kə'nɔmiks] *n.* 经济学
business ['biznis] *n.* 商业；事业
administration [ədˌminis'treiʃən] *n.* 管理，经营
extensive [iks'tensiv] *a.* 广泛的，广阔的
biological [ˌbaiə'lɔdʒikəl] *a.* 生物学的
genetics [dʒi'netiks] *n.* 发生学
technique [tek'ni:k] *n.* 技术；技巧
advantage [əd'vɑ:ntidʒ] *n.* 便利，利益
major ['meidʒə] *a.* 主要的；较多的
rudiment ['ru:dimənt] *n.* 基本原理
likely ['laikli] *a.* 象要……的
broaden ['brɔ:dn] *v.t.* 加宽
literature ['litəritʃə] *n.* 文学
utilize ['ju:tilaiz] *v.t.* 利用
simultaneous linear equation [ˌsiməl'teinjəs 'liniə i'kweiʃən] 联立一次(线性)方程
inventor [in'ventə] *n.* 发明家；创制者
analytic geometry 解析几何
founder ['faundə] *n.* 创立者
ultimately ['ʌltimitli] *ad.* 最终
belief [bi'li:f] *n.* 信念，信仰
untenable ['ʌn'tenəbl] *a.* 防守不住的，支持不住的
discipline ['disiplin] *n.* 学科，纪律
reducible [ri'dju:sibəl] *a.* 可化的，可约的
advent ['ædvənt] *n.* 出现
computer [kəm'pju:tə] *n.* 计算机；计算者
formidable ['fɔ:midəbl] *a.* 庞大的；艰难的
rewrite ['ri:rait] *v.t.* 再写，重写
identity matrix 单位(矩)阵
surprising [sə'praiziŋ] *a.* 惊人的，

奇怪的

occupy ['ɔkjupai] *v.t.* 占有

space [speis] *n.* 篇幅；空间

treatment ['tri:tmənt] *n.* 论述；处理；待遇

determinate [di'tə:minit] *a.* 确定的，一定的

unique [ju:'ni:k] *a.* 唯一的，独特的

transformation [ˌtrænsfə'meiʃən] *n.* 变换

rotation [rəu'teiʃən] *n.* 旋转

accomplish [ə'kɔmpliʃ] *v.t.* 完成

successive [sək'sesiv] *a.* 连续的

vector ['vektə] *n.* 向量

produce [prə'dju:s] *v.t.* 产生

product ['prɔdəkt] *n.* 乘积；产物

词 组

(to) play a ... role 起……作用

to ... advantage 便利；有利

注 释

① No matter what ... is 为一状语从句。 the student's field of major interest 为主语。what 为代词，作表语，原应在联系动词 is 的后面，因为语气强，形成局部倒装，移至句首。

24. THE GRAPHICAL SOLUTION OF LINEAR SYSTEMS

The construction of the graph of a single linear equation in two unknowns or of a linear system in two unknowns depends on several assumptions and definitions. These are:

I. To have at right angles to each other two lines, $X'OX$, called the x-axis, and $Y'OY$, called the y-axis.

II. To have a unit of distance on these lines.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the paper be the x-distance (or abscissa) of the point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the y-distance (or ordinate) of the point.

IV. That the x-distance of a point to the right of the y-axis be represented by a positive number, and the x-distance of a point to the left by a negative number, also that the y-distance of a point above the x-axis be represented by a positive number, and the y-distance of a point below the x-axis by a negative number. Briefly, distances measured from an axis to the right or upward are positive; to the left or downward are negative.

V. That every point in the surface of the paper corresponds to a pair of numbers, one or both of which may be positive, negative, integral, or fractional.①

VI. That of a given pair of numbers the first be the

measure of the x-distance and the second be the measure of the y-distance.

The point of intersection of the axes is called the origin.

The values of the x-distance and the y-distance are often called the coordinates of the point.

The relation between an equation and its graph may be stated as follows:

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point, the values of whose x-distance and whose y-distance satisfy the equation, is on the graph of the equation.

The graph of a linear equation in two unknowns is a straight line. Therefore it is necessary in constructing the graph of such an equation to locate only two points whose coordinates satisfy the equation and then to draw through the two points a straight line. It is usually most convenient to locate the two points where the line cuts the axes.② If these two points are very close together, however, the direction of the line will not be accurately determined. This error can be avoided by selecting two points at a greater distance apart.③

The graphical solution of a linear system in two unknowns consists in plotting the two equations to the same scale and on the same axes and obtaining from the graph the values of x and y at the point of intersection of the lines.

Through the graphical study of equations we unite

the two subjects of geometry and algebra, which seem very different, and learn to interpret problems of the one in the language of the other.

词　汇

linear system 线性系统
briefly [ˈbriːfli] *ad.* 简短地
upward [ˈʌpwəd] *ad.* 向上，在上面
downward [ˈdaunwəd] *ad.* 向下，在下面
error [ˈerə] *n.* 错误
apart [əˈpaːt] *ad.* 拆开地；独立地
unite [juːˈnait] *v.t.* 联合；合并

注　释

① ...one or both of which ... fractional 为一定语从句。
which 为关系代词，代替前面的 a pair of numbers.
one 和 both 均为代词，代替前面的 a pair of numbers, one 则代其中之一，而 both 则代两者。which 与 of 组成介词短语，作定语，修饰 one or both.

② ... where the line cuts the axes 为关系副词 where 所引导的定语从句，修饰主句中 the two points, where 在从句中仍作状语，修饰 cuts.

③ 介词短语 at a greater distance 作定语，修饰名词 points. apart 为副词，作状语，修饰 at a greater distance.

25. MATHEMATICS AS A LANGUAGE OF SCIENCE

A branch of science deals with a class of things, the changes in the members of the class, and the relations between these members. Thus the ideal form of a natural science is the same as that of mathematics. The objective in natural science is to discover relations which assert that when an event P is present in a situation, then the event Q also present. As a branch of science advances from a descriptive and qualitative stage to one where the relations can be expressed in a quantitative and explanatory manner, the science assumes a mathematical form. Astronomy at one time was a descriptive science, but work of Kepler and Newton established foundations, by means of which the laws of the motions of the heavenly bodies could be expressed mathematically. It is in this sense that mathematics is sometimes called the language of science.① When the postulates of a branch of science satisfy the requirements of the postulates of a branch of mathematics, then the hypothetical propositions of that branch and the deductions from them can be used in the verification of prediction of the propositions of the corresponding branch of science.

One difference between a scientific theory and its correlated mathematical system is that, if the mathematical deductions predict phenomena which conflict with experi-

ment, then some or all of the initial postulates of the scientific theory must be modified or discarded; but though the physical theory has failed, the mathematical system is not discredited, remaining as consistent as ever.② It has served one of its purposes in bringing to light the inadequacy of the scientific hypotheses.

At times, developments in mathematics have far exceeded the needs of any concrete science, properties of conic sections discovered by Appollonius, a Greek mathematician, were not applied until Kepler made use of ellipses to describe the motions of the planets around the sun. On the other hand, discoveries in science sometimes advance so rapidly that an adequate mathematical system lags behind. Important mathematical theories, developed as abstract sciences from apparently quite arbitrary sets of postulates, have later proved to be useful tools in applications of mathematics. It was from a study of algebraic equations that mathematician was led to predict that only 32 types of crystals would be found in mineralogy. Conical refraction of light was predicted by Hamilton from his mathematical study, before it was observed in the laboratory.

Mathematical deductions suggest experiments and also mathematical tools developed for the purposes of science have turned out to be powerful stimulus③ for growth in pure mathematics. The study of the flow of heat in a metal plate led the physicist Fourier to the invention of a series which not only solved complicated problems in the study of heat but also gave great impetus to the development of pure mathematics.

Science and mathematics advance in paralleled columns each assisting and stimulating the other④ — the hypothetical propositions of mathematics are called into play when the generalizations of science take on a quantitative form and frequently suggest new experiments; while, on the other hand, complexities in observed data of science stimulate the development of mathematics and broaden its foundation.

词 汇

ideal [ai'diəl] *a.* 理想的
natural ['nætʃrəl] *a.* 自然的
objective [ɔb'dʒektiv] *n.* 目的；任务；*a.* 客观的
assert [ə'sə:t] *v.t.* 断言；主张
event [i'vent] *n.* 事件
advance [əd'vɑ:ns] *v.t.*；*v.i.* & *n.* 前进
descriptive [dis'kriptiv] *a.* 描述的
explanatory [iks'plænətəri] *a.* 说明的；解释的
astronomy [əs'trɔnəmi] *n.* 天文学
Kepler ['keplə] 开普勒（人名）
Newton ['nju:tən] 牛顿（人名）
heavenly body ['hevnli 'bɔdi] 天体
mathematically [,mæθi'mætikəli] *ad.* 用数学；在数学上
hypothetical [,haipo'θetikəl] *a.* 假说的
proposition [,prɔpə'ziʃən] *n.* 定理
verification [,verifi'keiʃən] *n.* 证明，证实
prediction [pre'dikʃən] *n.* 预言
correlate ['kɔrileit] *v.t.* & *v.i.* 使互相关系
predict [pre'dikt] *v.t.* 预言
conflict [kənf'likt] *v.i.* 抵触
initial [i'niʃəl] *a.* 开始的，初期的；原先的
modify ['mɔdifai] *v.t.* 变更；限制
discard [dis'kɑ:d] *v.t.* 丢弃，废除
fail [feil] *v.i.* 失败；错误
discredite [dis'kredit] *v.t.* 怀疑
inadequacy [in'ædikwəsi] *n.* 不适当；不完全
hypothesis [hai'pɔθisis] *n.* 假设；臆说
(*pl.* hypotheses [hai'pɔθisi:z])
concrete [kɔn'kri:t] *a.* 具体的
lag [læg] *v.i.* 落后
behind [bi'haind] *ad.* 在后面
apparently [ə'pærəntli] *ad.* 显而易见地；表面上
later ['leitə] *ad.* 过后
crystal ['kristl] *n.* 结晶体，水晶
mineralogy [,minə'rælədʒi] *n.* 矿物学
conical ['kɔnikəl] *a.* 圆锥（体，形）的

refraction [ri'frækʃən] *n.* 折射
Hamilton ['hæmiltən] 哈密顿（人名）
stimulus ['stimjuləs] *n.* 刺激
growth [grəuθ] *n.* 发展
flow [fləu] *v.i.* 流动
heat [hi:t] *n.* 热
metal ['metəl] *n.* 金属
plate [pleit] *n.* 片、板
physicist ['fizisist] *n.* 物理学家
Fourier ['furiə] 傅里叶（人名）
invention [in'venʃən] *n.* 发明
impetus ['impitəs] *n.* 推动力
assist [ə'sist] *v.t.* 帮助
stimulate ['stimjuleit] *v.t.* 刺激
generalization [ˌdʒenərəlai'zeiʃən] *n.* 概括
complexity [kəm'pleksiti] *n.* 复杂性
datum ['deitəm] *n.* 数据（*pl.* data ['deitə]）

词　组

at one time 曾经
as ever 仍旧、照常
at times 时时
(to) lag behind 落后
(to) call into play 使动作，使活动
(to) take on 呈现

注　释

① It is ... that 是一种表示强调的句型结构。可以强调主语、宾语、状语等。凡是要强调的部分，均放在 It is 和 that 之间（即虚线部分）。本句用来强调 in this sense 这个状语部分。翻译时，语气较强，可译为：正是在这种意义上……。不强调时，本句应为：In this sense mathematics is sometimes called the language of science.

② 分词短语，remaining as consistent as ever 作状语，表示伴随动作。（有时也可表示结果、原因等）。

一般来说，分词短语如用逗号与句子的其它部分断开时，可作状语用。

③ 不定式短语 to be powerful stimulus 作表语。

此处 turned 相当于联系动词作用。

④ each assisting and stimulating the other 为独立（主格）分词结构，作状语。

26. THE CONCEPT OF FUNCTION, VARIABLE AND CONSTANTS

I

Seldom has a single concept played so important a role in mathematics as has the concept of function.① It is desirable to know how the concept has developed.

This concept, like many others, originates in physics. The physical quantities were the forerunners of mathematical variables, and relation among them was called a function relation in the late 16th century.

For example, the formula $s = 16t^2$ for the number of feet s a body falls② in any number of seconds t is a function relation between s and t. It describes the way s varies with t.③ The study of such relations led people in the 18th century to think of a function relation as nothing but a formula.

Only after the rise of modern analysis in the early 19th century could the concept of function be extended. In the extended sense, a function may be defined as follows: If a variable y depends on another variable x in such a way that to each value of x corresponds a definite value of y, then y is a function of x. This definition serves many a practical purpose④ even today.

Not specified by this definition is the manner of setting up the correspondence. It may be done by a formula as the 18th century mathematics presumed, but it can e-

qually well be done by a tabulation such as a statistical chart, or by some other form of description.

A typical example is the room temperature, which obviously is a function of time. But this function admits of no formula representation, although it can be recorded in a tabular form or traced out graphically by an automatic device.

The modern definition of a function y of x is simply a mapping from a space X to another space Y. A mapping is defined when every point x of X has a definite image y, a point of Y. The mapping concept is close to intuition, and therefore desirable to serve as a basis of the function concept. Moreover, as the space concept is incorporated in this modern definition, its generality contributes much to the generality of the function concept.

II

The numerical values of such physical quantities as time, length, area, volume, mass, velocity, pressure, temperature, etc., are determined by measurement. Mathematics deals with quantities divested of any specific content. From now on, when speaking of quantities, we shall have in view their numerical values. In various phenomena, the numerical values of certain quantities vary, while the numerical values of others remain fixed.[5] For instance, in uniform motion of a point, time and distance change, while the velocity remains constant.

A variable is quantity that takes on various numerical values. A constant is a quantity whose numerical values remain fixed. We shall use the letters X, Y, Z, ... etc., to

designate variables, and the letters *a, b, c*, ... etc., to designate constants.

In mathematics, a constant is frequently regarded as a special case of a variable whose numerical values are the same.

In considering specific physical phenomena it may happen that one and the same quantity in one phenomenon is a constant while in another it is a variable. For example, the velocity of uniform motion is a constant, while the velocity of uniformly accelerated motion is a variable. Quantities that have the same value under all circumstances are called absolute constants. For example, the ratio of the circumference of a circle to its diameter is an absolute constant:

$$\pi = 3.14159.$$

词　汇

seldom ['seldəm] *ad.* 很少，难得
desirable [di'zaiərəbl] *a.* 合乎需要的
originate [ə'ridʒineit] *v.t.* & *v.i.* 开始，发生
forerunner ['fɔː'rʌnə] *n.* 祖先；预报者
fall [fɔːl] *v.i.* 落下
(fell [fel], fallen ['fɔːlən])
vary ['vɛəri] *v.t.* & *v.i.* 改变
nothing ['nʌθiŋ] *n.* 什么东西也没有，无
rise [raiz] *n.* 兴起，上升
specify ['spesifai] *v.t.* 详细说明
presume [pri'zjuːm] *v.t.* 假定；猜想
equally ['iːkwəli] *ad.* 相等
tabulation [ˌtæbju'leiʃən] *n.* 作表；表
statistical [stə'tistikəl] *a.* 统计(上)的
chart [tʃɑːt] *n.* 图
description [dis'kripʃən] *n.* 描叙
typical ['tipikəl] *a.* 代表的，典型的
admit [əd'mit] *v.t.* 允许，承认
record [ri'kɔːd] *v.t.* 记录
tabular ['tæbjulə] *a.* 表的；扁平的
trace [treis] *v.t.* 跟踪
graphically ['græfikəli] *ad.* 写实

地；用图表

automatic [ˌɔːtəˈmætik] *a.* 自动的，机械的

device [diˈvais] *n.* 装置；器械

map [mæp] *v.t.* 画地图；*n.* 地图

image [ˈimidʒ] *n.* 影像，表像

moreover [mɔːˈrəuvə] *ad.* 此外，又

incorporate [inˈkɔːpəreit] *v.t.* 合并；编入

contribute [kənˈtribjuːt] *v.t.* 贡献；有助于

mass [mæs] *n.* 质量；许多；群众

velocity [viˈlɔsiti] *n.* 速度

pressure [ˈpreʃə] *n.* 压力

divest [daiˈvest] *v.t.* 脱去

content [ˈkɔntent] *n.* 内容；容积

happen [ˈhæpən] *v.i.* 发生

uniformly [ˈjuːnifɔːmli] *ad.* 同样地，一致

accelerate [ækˈseləreit] *v.t.* 加速；促进

circumstance [ˈsəːkəmstəns] *n.* 情况，环境

词　组

nothing but 只不过是

(to) admit of no 不容有……

(to) trace out 探寻踪迹；计划

(to be) divested of 被夺去，丧失

from now on 从此

in view 放在心里；作为目的

注　释

① 本句为主、从复合句。

主句：自 seldom … 至 in mathematics. 因为否定副词放在句首，谓语动词 has played 部分倒装，主语为 a single concept.

除 seldom 外，其它如 *only, not, never, hardly* 等副词开头的句子，因语气强，也影响句子词序，变为倒装句。

本课除本句外，第四段的第一句以 only 开头的句子也是这种倒装句的句型：

Only after the rise of modern analysis in the early 19th century could the concept of function be extended.

比较状语从句：由连接词 as 引导至句末为止，作状语，修饰 so 这句既是省略句，又是倒装句，其主语为 the concept of function. 谓语动词为 has played, played 省略，has 倒装。

② 本句为复合句。

主句：For example, the formula $s=16t^2$ for the number of feet s … is a function relation between s and t. 其中 for the number of feet s 是修饰 formula 的定语。

从句：a body falls in any number of seconds t 是一定语从句，修饰 the number of feet s，关系代词省略了，这个关系代词因为代表的是距离，所以在从句中作状语。

③ s varies with t 为定语从句，省略了 in which，which 为关系代词，代表前面的名词 the way。因“用……方法”的英语表达是 in ... way，所以知道 which 前应有介词 in。

④ many a practical purpose = many practical purposes。

⑤ fixed 此处为过去分词，作表语。remain 相当于联系动词。

27. FUNCTIONS

Continuous Variables. The quantities treated of in the Differential Calculus are regarded as capable of continuous variation in magnitude. Such a variable quantity may be represented by the distance, say x, of a point moving along a straight line from a fixed origin taken upon the line. The rate of variation of x is then represented by the velocity of the moving point, and the continuity of the variation means simply that x passes from one value to another, not suddenly, but gradually, so as to assume at some time each and every intermediate value of x.

When this motion is uniform, so that equal spaces are passed over in equal intervals of time, the velocity is constant, and the rate of x is said to be constant. Its measure is the space passed over, or increment received, in a unit of time; it is considered positive when x is increasing, and negative when x is decreasing.

At least two such variables will occur in any investigation; and if only two, one of them will depend upon the other for its value. The latter is called the independent variable, and is usually denoted by x. Denoting the other, or dependent variable, by y, it is said to be a function of x, and its dependence upon x is expressed by the general equation,

$$y=f(x).$$

When the relation is expressed by defiuite symbols as

$y=x^2$, $y=\sin x$, $y=\log x$, y is a known function of x and we may represent any value of x, together with the corresponding value of y, by the position of a point in a plane, as in analytical geometry. For this purpose, rectangular coordinates are used, the independent variable x being the abscissa and the function y being the ordinate of the point. A continuous variation in the value of x now produces a continuous variation in that of y, and this is represented by the motion of the point (x,y), or say the point P, in the plane. Thus, starting from the fixed point which has a certain value of x and the corresponding value of y, the moviug point P describes a line, straight or curved, in the plane. This line is called the graph of the function.

Take, for examples, the known function $y=x^2$, when $x=0$, $y=0$, and the representing point is at the origin.① Starting from this position, as x increases y increases, at first slowly, then more and more rapidly, and the moving point P describes the branch in the first quadrant of the parabola. Again, if, starting from any point of this branch, x decreases, the moving point returns along the curve and, after passing through the origin, describes the branch in the second quadrant. The complete parabola is thus the graph of the function $y=x^2$.

In considering the motion of the point P while describing the graph of a function, we shall, to fix the ideas,② suppose it such that the motion of R, the foot of the ordinate, (which we call the horizontal component of the motion of P) is from left to right at a uniform rate. In other words, x has a constant rate of increase. The graph

now shows at once whether during this motion the function, which is the ordinate, increases with x or decreases as x increases. In the first of these cases, it is called an increasing function of x, and in the second a decreasing function. It should be noticed, of course, that the oblique motion of P itself③ is not here considered. But, if we choose to project P upon the axis of y by a parallel PS to the axis of x, then the motion of S will indicate the increase or decrease in y with which we are concerned. If now the function is an increasing one, S will be moving upward as R moves to the right, and P will be moving obliquely upward. The graph is then said to have a positive gradient or slope. A positive slope thus means an increasing function, and a negative slope a decreasing function.

A function may be an increasing one for a certain range of values of x, and a decreasing one for another range of values. For example, the graph of x^2 shows that x^2 is an increasing function for positive, and a decreasing function for negative values of x. By drawing the graph of $\log x$, or curve whose equation is $y = \log x$, the student will see that the curve gradually illustrates the fact that $\log x$ is always an increasing function of x. Again, the curve $y = \sin x$ shows that $\sin x$ is alternately an increasing and a decreasing function.

词　汇

treat [tri:t] *v.t.* 讨论；待遇

differential calculus [ˌdifəˈrenʃəl ˈkælkjuləs] 微分

capable [ˈkeipəbl] *a.* 能……的，有能力的

variation [ˌvɛəriˈeiʃən] *n.* 变化；变量

along [əˈlɔŋ] *prep.* & *ad.* 沿；一块

儿

rate [reit] *n.* 速度，比率

continuity [ˌkɔntiˈnjuːiti] *n.* 连续，连合

pass [pɑːs] *v.i.* & *v.t.* 通过；前进 (passed; passed, past)

suddenly [ˈsʌdnli] *ad.* 突然地

intermediate [ˌintəˈmiːdjət] *a.* 中间的

interval [ˈintəvəl] *n.* 间隔

increment [ˈinkrimənt] *n.* 增量

investigation [inˌvestiˈgeiʃən] *n.* 调查，研究

independent variable 自变量

dependent variable 因变量

rectangular [rekˈtæŋgjulə] *a.* 矩形的，直角的

component [kəmˈpəunənt] *n.* 成分；*a.* 组成的

oblique [oˈbliːk] *a.* 倾斜的

project [prəˈdʒekt] *v.t.* 投影；画投影线

decrease [ˈdiːkriːs] *n.* 减小

obliquely [əˈbliːkli] *a.* 倾斜地

gradient [ˈgreidiənt] *n.* 梯度

slope [sləup] *n.* 斜面，斜坡，坡度

alternately [ɔːlˈtəːnitli] *ad.* 交替地

词　组

(to) treat of 讨论

capable of 能……，可以……的

in other words 换言之

注　释

① $y=x^2$ 是 the known function 的同位语。合起来作 take 的宾语，for examples 为宾语补足语。之所以倒装是因为宾语比宾语补足语长，按照英文的习惯排列是短的在前，长的在后。可译为：以（或拿）……作为例子。

② to fix the ideas 是插入语。

③ itself 是返身代词，此处用作加强语气。它的另一用途是作宾语。

28. THE DEVELOPMENT OF THE NUMBER SYSTEM (I)

The development of the number system through generalization of the number concept is one of the instructive studies in mathematics. The term number originally meant the integers, called natural numbers in contrast with the various types known as artificial numbers which have been included in the number system as it has expanded.

Integers. The number system began when mankind distinguished between classes of things without the necessity of describing them, i.e., between five fish and the number five. Even the most primitive tribe has words for at least some number concepts. A considerable development of the system of integers in very ancient times is indicated by the cuneiform records of the Babylonians which contain tables of squares, cubes, and some progressions.

Ancient systems of notation for integers were clumsy, and in some instances, calculations were performed by means of assisted and recorded computations. In time this was replaced by the abacus, an instrument consisting of a frame with counters sliding on wires or in grooves which is still in use today in China, Japan, Russia, India, and other oriental countries.

Fractions, the next types of number, arose through the practical need of subdividing possessions. Fractions

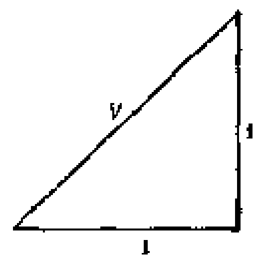
such as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ with unit numerators were sufficient for simple subdivisions. The Ahme papyrus, 1400 B.C., describes methods of transforming fractions such as $\frac{3}{4}$ and $\frac{5}{7}$ to fractions with numerators of unity. Thus

$$\frac{3}{4}=\frac{1}{2}+\frac{1}{4} \text{ and } \frac{5}{7}=\frac{1}{2}+\frac{1}{7}+\frac{1}{14}$$

The addition of $\frac{3}{4}$ and $\frac{5}{7}$ could then be performed by combining those simple fractions. In practical problems of commerce, computations involving fractions were simplified by the use of tables involving subunits. The minutes and seconds measured by our clocks are survivals of the sexagesimal fractions of the Babylonians.

Irrationals. The third stage in the development was the discovery of line segments which cannot be represented by integral or fractional multiples of a line segment chosen as a unit length. This discovery was made in connection with a study of right triangles. The theorem of Pythagoras states that the sum of the squares on the two sides of a right triangle is equal to the square on the hypotenuse.

Greek mathematicians knew right triangles in which this relation was expressed by integers, for instance, the right triangles with sides 3, 4, 5 and 5, 12, 13. If, however, the right triangle is isosceles, they recognized that the length of the hypotenuse is *not* commensurable with either side. For if the length of the sides is taken as the unit, then the hypotenuse is given by the equation



$$V^2 = 1^2 + 1^2 = 2. \tag{1}$$

There is no integral value of V which will satisfy the equation. If there were a fraction $\frac{a}{b}$ where a and b are integers, then

$$\frac{a^2}{b^2} = 2 \text{ or } a^2 = 2b^2. \tag{2}$$

We can assume that a and b have no common factors, for such common factors could be canceled out to begin with.① From (2) it follows that a^2 is an even number, and therefore a must be even, say, $a = 2a$. Substituting the value of a in (2), gives $4a'^2 = 2b^2$, and hence

$$b^2 = 2a'^2$$

Therefore, b^2, and hence b, is an even number. It follows then that a and b have the factor 2. But this contradicts the hypothesis that a and b have no common factor. Thus the assumption that the hypotenues can be represented by a fraction $\frac{a}{b}$ leads to a contradiction and is, therefore, false.

This "indirect proof" shows that the symbol $\sqrt{2}$ cannot correspond to any integral or fractional number, and proves the existence of magnitudes which cannot be represented by either of these types of numbers.

Zero. The next advance came in India (about 600 A.D.) through the invention of the symbol 0 to represent, originally an empty column occurring in a calculation on an abacus.

The introduction of zero and the invention of the principle of position, whereby② the same number symbol was used in different senses according to the position it

occupied in a number, improved the notation of number to such an extent that calculations could be performed more rapidly with symbols than with the abacus. Through the Middle Ages the debate between the advocates of the two methods of calculation continues with as much intensity as the controversy in our time concerning the respective merits of the metric and English systems of units.

词 汇

instructive [ins'trʌktiv] *a.* 有教益的，有启发的
originally [ə'ridʒənəli] *ad.* 本来，最初
contrast ['kɔntræst] *n.* 对照
artificial [ˌɑːti'fiʃəl] *a.* 人为的
include [in'kluːd] *v.t.* 包括
mankind [mæn'kaind] *n.* 人类
distinguish [dis'tiŋgwiʃ] *v.t.* & *v.i.* 区别
necessity [ni'sesiti] *n.* 需要
fish [fiʃ] *n.* 鱼
cuneiform ['kjuːnifɔːm] *a.* 楔形
Babylonian [bæbi'ləunjən] *a.* 巴比伦人
progression [prə'greʃən] *n.* 级数
clumsy ['klʌmzi] *a.* 笨拙的
replace [riː'pleis] *v.t.* 替代
abacus ['æbəkəs] *n.* 算盘
instrument ['instrumənt] *n.* 器具
frame [freim] *n.* 框子
counter ['kauntə] *n.* 珠
slide [slaid] *v.i.* 滑动 (slid [slid], slid)
wire [waiə] *n.* 线
groove [gruːv] *n.* 沟
Japan [dʒə'pæn] *n.* 日本
Russia ['rʌʃə] *n.* 苏联
India ['indjə] *n.* 印度
oriental [ɔːri'entl] *a.* 东方的
possession [pə'zeʃən] *n.* 财产
numerator ['njuməreitə] *n.* 分子
Ahme （人名）
papyrus [pə'paiərəs] *n.* 纸草（古埃及人等用来制纸的一种水草）；水草制的纸（*pl.* papyri [pə'paiərai]）
commerce ['kɔməs] *n.* 商业
simplify ['simplifai] *v.t.* 简化
subunit ['sʌbjunit] *n.* 小单位
survival [sə'vaivəl] *n.* 残余
sexagesimal [ˌseksə'dʒesiməl] *a.* 六十分的
stage [steidʒ] *n.* 阶段
theorem ['θiərəm] *n.* 定理；原理；法则
Pythagoras [pai'θægəræs] *n.* 毕达哥拉斯（人名）
commensurable [kə'menʃərəbl] *a.* 可度量的

cancel ['kænsl] *v.t.* 约简
even number 偶数
substitute ['səbstitjut] *v.t.* 代替
contradict [ˌkɔntrə'dikt] *v.t.* 与……矛盾
A.D. (Anno Domini) 公元
empty ['empti] *a.* 空的
whereby [wɛə'bai] *ad.* 借助于它
the Middle Ages 中世纪
debate [di'beit] *n.* 争论
advocate ['ædvəkeit] *n.* 支持者
intensity [in'tensiti] *n.* 强烈
controversy ['kɔntrəvəːsi] *n.* 争辩
concerning [kən'səːniŋ] *prep.* 关于
merit ['merit] *n.* 优点
metric system ['metrik 'sistim] 公制、米制
English system 英制

词 组

in contrast with 与……对照
(to) have words 有记载；谈到
in time 后来；逐渐
in use 应用
in connection with 与……关连
to begin with 首先
it follows that ... ……因此……
to an extent 到某种程度

注 释

① to begin with 为不定式短语，作状语，修饰 could be canceled out。
② whereby 为关系副词，引导定语从句，修饰 the principle of position。

29. THE DEVELOPMENT OF THE NUMBER SYSTEM (II)

Negative Numbers. The fifth stage in the evolution of the number concept was also due to Hindu mathematicians and resulted from the study of algebraic equations. Equations of the form,

$$x+a=b,\ x^2=a$$

were classified as possible or impossible according as the unknown did or did not represent a uumber already included among the accepted numbers.

Thus $x+3=5$ was a possible equation because its root was the integer 2, while $x+5=3$ was an impossible equation as

$$x=-5+3=-2$$

was not a known number. Such numbers, called negative, were finally accepted when it was observed that they could be used to represent a debt. Thus the concept of direction was added to that of magnitude in this extension of the number concept. From the concept of negative numbers evolved the theory of directed magnitudes known as vector analysis which has become a powerful tool in mathematical physics.

Rational Nnmbers. The totality of positive and negative integers aud fractions and zero is called the class of rational numbers.

The sum and the product of two integers are always

integers. This property is expressed by saying that[1] the class of integers is closed with respect to the operations of addition and multiplication.

The inverse operations, subtraction and division, performed on integers do not always yield integers. The desire to eliminate exceptions brought about the construction of the rational class of numbers, all of which can be obtained from unity by means of the rational operations, addition, multiplication, subtraction, and division. The rational numbers may be represented graphically by means of points of a straight line. Two arbitrary points are chosen to represent *zero* and unity, and the distance between these points is used to determine a scale by means of which a point on the line can be determined for every rational number, positive or negative.

In every interval of the number axis, no matter how small, there are always rational numbers, which property is expressed in the statement; the set of rational numbers is everywhere dense.

Real Numbers. In spite of the property of density, the rational numbers are not sufficient to represent every point on the axis. The irrational number $\sqrt{2}=1.414\ldots$, corresponds to a point not represented by any number of the rational class. It can be shown that to every point on the number axis can be assigned a decimal which may be finite or infinite and if infinite, periodic or not periodic.[2]

The rational numbers can be represented by finite decimals or periodic infinite decimals as, for example

$$\frac{1}{8}=0.125, \text{ and } \frac{1}{3}\cdot\frac{0}{3}\cdot\frac{4}{3}=0.312312312\ldots,$$

while the irrational numbers are represented by infinite decimals which are not periodic.

Any point P on the axis, which does not correspond to a rational number will correspond to an infinite non-periodic decimal

$$a+0.a_1a_2a_3\ldots$$

which is said to be the real number corresponding to the point P.

The irrational numbers are divided into two classes algebraic and transcendental irrationals according as they are roots of algebraic equations or not. Examples of transcendental irrationals are, the ratio of the circumference of a circle to its diameter, 3.14159 and the base of the natural system of logarithms, $e=2.718\ldots$.

The number $2^{\sqrt{2}}$ has recently been proved to be transcendental. The rational and the irrational number together constitute the class of real numbers.

词　汇

due [dju:] *a.* 由于,到期
Hindu [ˈhinˈdu:] *a.* 印度的
classify [ˈklæsifai] *v.t.* 分类
impossible [imˈpɔsəbl] *a.* 不可能的
debt [det] *n.* 债务;亏欠
evolve [iˈvɔlv] *v.t.* 演化;开展
yield [ji:ld] *v.t.* 产生
eliminate [iˈlimineit] *v.t.* 消除
dense [dens] *a.* 浓密的
density [ˈdensiti] *n.* 稠密度,密度
periodic [ˌpiəriˈɔdik] *a.* 周期性的
transcendental [ˌtrænsenˈdentəl] *a.* 超越的

词　组

(to be) due to 由于
according as 依照,据
with respect to 关于
no matter how 无论如何
(to) correspond to 对应,符合

注　释

① 此处 that 为连接词,引导动名词 saying 的宾语从句。

② It 是先行代词,作形式主语,代表后面连接词 that 引导的主语从句。主语从句是完全倒装句。主语是 a decimal, 介词短语 to every point on the number axis 作状语,修饰谓语动词 can be assigned。

本句之所以倒装是因为:一方面将 every point 倒在前面,和上面讲的 point 接近;另一方面,将 a decimal 倒在后面,便于为关系代词 which 所引导的定语从句所修饰。

自 and 至句末为省略句。完全形式应为:

and which may be periodic or not periodic if it is infinite.

30. THE DEVELOPMENT OF THE NUMBER SYSTEM (III)

Complex Numbers. While the irrational numbers originated in geometry,① the negative and the complex numbers are of algebraic origin.

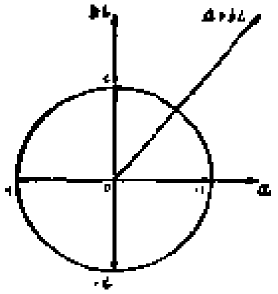
Diophantos (300 A. D.) solved quadratic equations in much the same way as is done at present, obtaining the solution in a form equivalent to $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. At first the solution was regarded as possible only when the number under the radical was a perfect square and the root belonging to the positive sign was alone recognized.

The Hindu Bhaskara (1114 A.D.) recognized the double sign of the square root, reckoned with irrationals, but considered the square root of a negative number as impossible.

Cardan, in 1545, published an algebraic solution of the cubic equation due to Tartaglia and showed by substitutions and formal manipulation that a complex number of the form $a+b\sqrt{-1}$ satisfied an equation, but he did not accept such numbers because he knew of no interpretation that could be attached to them.

A geometric representation of an imaginary number was proposed by Argand in 1806.

In order that the relation $\sqrt{a}\cdot\sqrt{a}=a$②, where a is



a positive number, might hold also for negative numbers, it was assumed that $\sqrt{-1}\cdot\sqrt{-1}=-1$. This relation written in the form $\frac{-1}{\sqrt{-1}}=\frac{\sqrt{-1}}{-1}$ indicates that $\sqrt{-1}$ is a mean proportional between 1 and -1. Using the construction for determining a mean proportional between two line segments, a geometric representation of the symbol $\sqrt{-1}$ is obtained by constructing a circle of unit radius with center at the origin of the axis of real numbers and erecting a perpendicular at 0 intersecting the unit circle. This directed line segment of unit length,[8] perpendicular to the segment representing 1 Argand took to represent $\sqrt{-1}$ which is usually symbolized by i.

Imaginary numbers such as bi are represented by points on a line perpendicular at the origin to the axis of real numbers.

Further progress was made by associating complex numbers with the points of a plane. Thus a complex number $a+bi$ is represented by the point P whose Cartesian coordinates are (a,b).

An alternative representation of the complex number $a+bi$ is the directed line segment extending from the origin O to P.

With this geometric interpretation equations such as $x^2=-1$ and $x^2-x-1=0$ were no longer regarded as impossible as their roots $\pm\sqrt{-1}$, $\frac{+1\pm\sqrt{-3}}{2}$ could be as

easily represented as if[4] they were real.

Complex numbers are particularly adapted for the expression of relations in two dimensions. Accepted[5] originally to satisfy the desire of mathematicians to explain so-called impossible solutions of equations, they have become indispensable in mathematical physics for the study of the flow of electricity, magnetism, or heat in a metal plate. They are also used in the theory of map making and in hydrodynamics. Imaginary numbers have ceased to be imaginary except in name.

The number system of the algebra of complex number is classified as follows:

- Complex
  - Real
    - Positive
      - Rational
        - Integers
        - Fractions
      - Irrational
        - Algebraic
        - Transcendental
    - Negative
  - Imaginary

The complex number system is said to be closed with respect to the six fundamental operations, for if any of these operations are applied to complex numbers, no new type of number arises.

Each generalization of the number concept brought changes in the meanings of the operations. The symbol $+$ and $\times$ have different meanings in the addition and multiplication of complex numbers from what[6] they have in operating on integers, rationals, or irrationals.

Multiplication of Two Complex Numbers. The product $A \times B$ is obtained geometrically by rotating the vector B through the same angle and changing its length in the same

proportion that the unit vector OI must be rotated and extended to be brought into coincidence with OA. This is easily performed by making the triangle OBP similar to the triangle OIA.

Throughout the generalization of the numbers and operations, the associative, commutative, and distributive laws of addition and multiplication remained unchanged.

Further development of the number concept was brought about through changes in the fundamental postulates of algebra. Weierstrass proved that it is impossible to construct a class of numbers more general than the complex numbers if all the postulates are retained without change.

词 汇

Diophantos 丢番图(人名)
perfect [ˈpəːfikt] *a.* 完全
Bhaskara 巴斯卡拉(人名)
reckon [ˈrekən] *v.t.* 计算
Cardan 卡尔顿(人名)
publish [ˈpʌbliʃ] *v.t.* 发表
cubic [ˈkjubik] *a.* 立方的,三次的
Tartaglia 塔尔塔格里亚(人名)
manipulation [məˌnipjuˈleiʃən] *n.* 操作
propose [prəˈpəuz] *v.t.* & *v.i.* 计划;提出
Argand 阿干德(人名)
mean proportional [miːn prəˈpɔːʃnl] 比例中项
erect [iˈrekt] *v.t.* 建立,安
symbolize [ˈsimbəlaiz] *v.t.* 用记号(符号)表示
Cartesian Coordinates [kaːˈtiːziən kəuˈɔːdnits] 笛卡儿座标
alternative [ɔːlˈtəːnətiv] *a.* 二中择一的
directed [diˈrektid] *a.* 有向的
particularly [pəˈtikjuləli] *ad.* 特别
dimension [diˈmenʃən] *n.* 向度
explain [iksˈplein] *v.t.* 解释
indispensable [ˌindisˈpensəbl] *a.* 必要的,不可少的
magnetism [ˈmægnitizm] *n.* 磁性
map making 制图
hydrodynamics [ˈhaidrədaiˈnæmiks] *n.* 流体力学
cease [siːz] *v.i.* & *v.t.* 停止,不再
rotate [rəuˈteit] *v.i.* & *v.t.* 旋转
coincidence [kəuˈinsidəns] *n.* 相同,一致

unchanged [ʌn'tʃeindʒd] *a.* 不变的

Weierstrass 维尔斯特拉斯(人名)

词 组

much the same 差不多相同

in order that 为了……起见

in name 名义上

注 释

① while 是连接词,放在句首,引导让步状语从句,译为虽然。

② In order that 为复合连接词,引导一目的状语从句。主语为 the relation $\sqrt{a}\cdot\sqrt{a}=a$. $\sqrt{a}\cdot\sqrt{a}=a$ 为 the relation 的同位语。此状语从句的谓语动词为 might hold. 主语和谓语动词之间为关系副词 where 所引导的定语从句(修饰$\sqrt{a}\cdot\sqrt{a}=a$) 所隔断。

③ This directed line segment of unit length 为及物动词 took 的宾语,之所以倒装在句首是因为一方面使它接近上面所谈到的 line segment; 另一方面还因为它后面有形容词短语 perpendicular to the segment representing 1 修饰它。这样倒装,比较合适。

④ as if 为复合连接词,引导方式状语从句,用虚拟语气。

⑤ accepted 为过去分词,引导分词短语到 solutions of equations 为止,作状语,修饰句中谓语 have become indispensable。

⑥ what 为连接代词,引导介词 from 的宾语从句。what 在从句中作及物动词 have 的宾语。what 相当于 the meanings which。

Chinese Translations of the Text

译 文 目 录

1. 数学来自实践

恩格斯说过:“数学像所有别的科学一样，起因于人们的需要”。从一开始,数学就直接或间接地试图满足生产中的一定的需要。

人类在社会实践中开始感到数东西和计算容器体积的需要。从这种早期的需要,就产生了数和形的概念。后来，几何(学)因量度土地的问题而得到发展,三角(学)则来自测量问题。为了使计算更为简单，人类还学会了使用符号,结果就产生了代数(学)。

在初等数学中，我们只探讨(研究)常数。随着17世纪工业的迅速发展,用常数来计算已不再能够满足生产的需要。生产中的许多新问题要求得到解决。为了解决这些问题，人类开始研究变(化的)量和运动。这种从常数到变量的跳(飞)跃，产生了数学的一门新的分支——微积分学。

总之,数学来自人类的社会实践。在研究数学时，我们必须理论联系实际。我们必须使数学为我国的社会主义革命和社会主义建设服务。

2. 数学的语言

数学的语言就是由记号和符号组成的语言。全世界都一样。

一些最有名的数学符号就是阿拉伯数字1,2,3,4,5,6,7,8,9,0及加、减、乘、除、等式记号。

阿拉伯数字之所以得到使用是因为它们很方便。在111这个数中，使用了三个1,并且每个1都有不同的含义。最右边的1代表了一这个数。从右边起第二栏中的1代表了十这个数。在第三栏中的1代表了一百这个数。

如果我们想要写三个十这个符号， 我们就将一个3写进从右边起第二栏中。但是,我们认不出它是第二栏;我们应当在第一栏中写下个

什么东西。这样，就有必要认为三个十不加上什么东西，因而使用一个符号来代表没有东西的那一栏（空栏）。为此目的，我们使用了0这个符号，并称它为零。零加任一数仍等于该数，零乘任一数等于零。

3．度 量

在自然科学的发展中，在人类开始将结论建立在实验的事实而不是在推论的基础上之后，我们观察到科学上的成就迅速增加。然而，实验证明了对自然界的某个方面进行了量的研究。而这样一种研究的重要部分就是对于它所研究（探讨）的事物进行量度。量度任何量意味着（就是）将它与公认为标准的单位相比较，并找出它比标准单位大多少倍或小多少倍。一物体的长度是通过求出它是某标准单位长度的多少倍长来度量的。例如，如果拿这本书作为一个标准，将其沿着一书桌表面头尾相接地量上五次，我们就知道该书桌有五本书那么长。如果将这本书头尾相接地放下五次而尚未完全达到书桌另一端时，我们说，它的长度稍许比五本书多一点。在科学工作上，这种“稍许超过”的部分并不够准确。为了更为准确起见，我们必须量度出书桌超出了五本书长度的多少分之一。如果我们量度出该书桌比五本书长还要长五分之一本书，我们说，它的长度为五又五分之一或五点二本书长。更为准确的量度则是将书再分为十等分。我们就会量度出书桌要比五点二本书的长度稍为长一点。我们又不得不量度一下通过细分后书桌长出五点二本书的部分。如果我们发现该分数的部分为细分的二分之一，我们就会写下书桌的长度为五点二五本书。最后的量度的准确度显然远远超过那些对较大单位的量度的准确度。需要的准确度越大，则细分就必然分得越小。

一物体的重量同样是由（根据）发现它比某被公认（接受）的标准重量单位重多少而决定的。例如，如果一块铜的重量为一标准磅的四倍，则它的重量为四磅。同样，标准重量分得越细，称就称得越准确。

4. 古代中国数学家对圆的量度

我们已经知道古代中国人用三来作为 π 的值或他们按三比一来计算圆周和直径之比。π 的值至少早在公元前 12 世纪就在中国使用了。但是中国人对这个不精确的 π 值决没有感到满足。从那时起，人们就作了很大努力来改进它的准确度，并且取得了光辉的成就。

在最早的中国求圆者中，必须首先提到张衡这个人。张衡是汉朝的一位著名学者。虽然张衡所计算出来的 π 值，以圆的圆周的平方对外切正方形的周长的平方之比为五比八的形式载入"九章算术"，但是，他对圆的计算方法已经失传。这等于取 π 为 $\sqrt{10}$.

在三国时期，还有一位数学家叫刘徽。在他所作的"九章算术"注解中，我们发现他(对圆)的积圆法的细节。

刘徽对圆的度量是从圆的内接六边形开始的。该圆的直径取作两英尺。六边形的每边等于圆的直径的一半。在这个六边形上，刘徽通过将它的边数加倍而画出一个十二边形。然后再将边数加倍而画成一个二十四边形，等等。按这种方法，这样形成的多边形的面积逐渐趋近于圆的面积，两者的差别一步步地缩小。

刘徽以后两个世纪时又出现了另一位也更著名的求圆者祖冲之。他发明了一种比他的前人所曾沿用过的更为有效的方法，从而获得准确的 π 值。

π 为 $\frac{355}{113}$。从这一点可以看到中国先于欧洲一千三百多年就已具有 π 的准确的值了。在那里(欧洲)，同样的值是于 1855 年求得的。

祖冲之于公元 500 年以 71 岁高龄逝世。他的儿子祖暅之是继他的父亲之后的又一著名数学家。就是他第一个推导出球体体积世界通用的公式，即球体体积等于 $\frac{1}{6}\pi D^3$，其中 D 表示直径。

5. 为什么我们要按十个一组地来计数东西?

为什么我们要按十个一组地来计数东西呢?理由是我们有十个指头。很久以前，当人们不得不计数许多东西时，他们把东西跟他们的手指相比，起初，数出和他们两手的手指数目相等的东西。然后，他们将这些东西放在一旁成一组。如果有十个以上的东西要数，他们就编成了更多的组。我们可以将我们的数字称为两手数字，因为这些数字是由于在两只手上数东西产生的。

也有些人用一手数字。因为一只手上有五个手指，他们就用五个一组来数东西。两千多年以前，住在意大利的人使用过一手数字。我们将他们书写的数字称为罗马数字，这种数字甚至今天我们还用。在罗马数字中，I 代表一，V 代表五。要写六，罗马人写成 VI，它意味着（指的是）五加一。很久以前，那时人们不穿鞋子，他们还能用脚趾(指)来计数。因此，一些人有一种赤脚算术。他们按二十个一组来数出东西。

有时，人们用十二个一组来数东西。为了某些目的，我们仍然用十二个为一组的体系。当我们用钟来计算时间时，我们从一到十二来计算小时，然后再由一从头到尾再来一遍。十二个一组的体系还在英语数字名称上留下痕迹。在英语中，从一到十二每个数都有一个词。给十二以上的数字取名时则使用十个一组的体系。

成千上万年以前有一个部落习惯于用六十个一组来数东西。我们(今天)计算时间时，仍然使用按六十个一组的体系。一分等于六十秒，一小时等于六十分。

6. 古代的计算和测量

如果我们能够回到千万年以前的话，我们就会发现生活在当时的人不得不老在考虑到必要的衣食供应和适宜的住所。因此，甚至原始人都被迫考虑如下的问题：有多少人一定得供养?一季中我们需要多少

衣食？我们的供应品能维持多久？

显而易见，要回答这些问题，人们必须计算和测量。

在古代，人们不能像我们现在这样来进行计算。最初，所有的计算都是用小石子，棍棒，贝壳或其他方便的东西来进行的。但是，随着时间的流逝，人们学会了用（他们的）手指来进行计算；并且，由于我们都有十个手指，因此，十这个数就成为世界各地全部计算的基础。这种十进位的计算制逐步地导致我们目前读数和写数的方法，并且产生了现在称为算术的这一数学分支。

原始人也不能像我们这样测量。他们一点不懂得直尺和码尺；而是代之以步测距离或长度，或者他们用手来进行较小的测量。人们一旦习惯于农业耕作后，就产生了更为准确的计算方法。自然，人们想要知道他们的农田有多大，能种多少食物，谷仓里能贮藏多少食物以及井里能装多少水。这些和许多类似的问题，由于发明了简单而实用的法则而逐步得到了解决。所有这种知识为现在称之为几何（学）的这一数学分支作好了准备。

7. 方　程

方程是表明两个相等的数或数的符号相等的式子。

因此，$r(r-5)=r^2-5r$ 和 $x-3=5$ 都是方程式。

方程有两种：恒等式和条件等式。

算数或代数恒等式是方程。在这样一个方程中，不是两端边相同，就是在进行所示的运算后变得相同。

因此，$12-2=2+8$；$(m+n)(m-n)=m^2-n^2$ 都是恒等式。

包含字母的恒等式对于式内字母的任何一组数值都是适合的。

因此，在假如 $x=3$，$r=7$ 时，恒等式 $x(r+2)=xr+2x$ 则成为 $3(7+2)=21+6$ 或 $27=27$。

如一等式只对其中一个字母的某些数值或对其中两个或两个以上字母的某些组相应数值适合的话，则此等式为条件等式或简称方程。

因此，$3x-5=7$ 只对 $x=4$ 是适合的；$2x-y=10$ 对 $x=6$，$y=2$

以及 x, y 的其他许多对数值都是适合的。

方程的根是能满足该方程的任何数或数字符号。

求方程的根就叫做解方程。

方程有很多种，如一次方程、二次方程等等。

8. 方程的用处

如果把任意两组项或多项式写成为一组等于另一组的等式时，这样的表达式称为方程。

方程的用处很大。我们可用方程来解求一未知数，即我们能容易地找到一种方法来把那个未知数放在我们能发现它的位置上。这样做，就叫做解方程。让我们从很简单的方程开始，试图找出对它们进行运算的方法。我们能将方程用于许多算术的问题。我们或许注意到几乎每个问题都给了我们以一种或多种表示某物和某物相等的说明：这就给出了方程，如果我们需要的话，我们就可以进行运算。

解方程就是找出未知项的值。要做到这点，当然，我们必须移项，直到使未知项单独处于方程的一边为止，这样一来，就使得它等于方程另一边的那些项。然后，我们就得到未知数的值，也就是问题的答案。因此，解方程意味着进行移项，而不使方程失去其真实性，直到方程的一边（不论哪一边）只留下一个未知数时为止。

9. 根　式

根式是一个代数或算术表达式的一个确定的根。

因此，$\sqrt{4}$，$\sqrt{7}$，$\sqrt[3]{5}$ 和 $\sqrt{r^2-3r-1}$ 都是根式。

像在根式 $\sqrt[3]{5}$ 中小的数字3，称为根式的指数。

指数决定根式的次数，并指明要开方的根。

被开方数是在根号下面的数字或式子。在根式 $\sqrt{8}$ 和 $\sqrt[3]{5am}$ 中，被开方数分别为8和 $5am$。

根式可用两种方法中任一种书写：用根号或用分数指数。

因此，$\sqrt{7}$和 $7^{\frac{1}{2}}$意义相同。$\sqrt[3]{c^2}$ 等于 $c^{\frac{2}{3}}$ 等等。

有理数是一个正的或负的整数，或者是能表示为这样两个整数的商或比的任何数。

因此，8，-4，$\frac{3}{5}$或 6.713 均为有理数。

不为有理数的任一实数是无理数。

如果在根号下的一数其所示的根不能确切得到，那么，该根式代表了一个无理数。

例如：$\sqrt{5}$和$\sqrt[3]{7}$ 都是无理的。

循环小数，虽然是循环不已，但却并不是无理数，因为任一循环小数都能用普通分数来表示，因此，是有理的。

负数的平方根叫做虚数。

于是，代数里的一切数均可归入以下两类中的任一类：实数和虚数。

如我们所知，实数有两类：有理数和无理数。

不尽根数是无理数，在其中，被开方数是有理的。

10．无理方程

含有一个未知数的无理方程就是该未知数出现在根式下面，或者受到分数指数影响的这样一个方程。

因此，$x^2-2\sqrt{x}+1=0$ 和 $x^{\frac{1}{4}}-x^{\frac{1}{2}}+1=0$ 和 $y-(2y)^{\frac{1}{2}}-4=0$ 均为无理方程。

解这类方程所涉及到的一个困难来自以下事实：有时所得到的结果并不满足所给定的方程，因此，不是此方程的根。这类的根称为奇根。

例　　题

(a) 解　$\sqrt{x-3}-5=0$　　　　(b) 解　$-\sqrt{x-3}-5=0$

解：移项，

$\sqrt{x-3}=5 \quad (1)$	$-\sqrt{x-3}=5 \quad (1)$
开方，$x-3=25 \quad (2)$	$x-3=25 \quad (2)$
解: $x=28$	$x=28$
验算: $\sqrt{28-3}-5=0$	$-\sqrt{28-3}-5=0$
$\sqrt{25}-5=0$	$-\sqrt{25}-5=0$
$5-5=0$	$-5-5=0$
此式是正确的。	此式是不正确的。

对这些解进行研究发现，两个(1)式之不同仅在于它们左端前面的符号。这种差别在平方后便消失了，因面，两个方程(2)是等同的。在(a)和(b)两者之中往下作时，都是在于对(2)求解。其所得的结果确为此方程的根。究竟所求得的根满足(a)和(b)两者，或者只满足二者之一，只能用代入法来决定。在这种情况下，很明显，(a)是一个等式，而(b)却不是，它仅仅是一种以方程的形式出现的错误的式子。

总之，原方程的全部的根肯定是在所求得的结果中，但如有任何结果不能满足原方程，它就不能叫做根。这意味着所有的结果必须进行验算。

在无理方程中，如在直到目前为止的一切(运算)工作中，大家共同理解，除非一个很式或受到分数指数影响的式子前面有双号±，否则，它仅只有一个值，(这情况)就正如任何其它数字符号一样。

因此，$\sqrt{81}$ 意指$+9$而不是-9。

同样，$9^{\frac{1}{2}}$ 意指$+3$而 $-9^{\frac{1}{2}}$ 意指$-\sqrt{9}$，或-3，$x^{\frac{1}{2}}$ 意指 $+\sqrt{x}$，而不是$-\sqrt{x}$。

如果我们牢记这一事实，观察一下上面的(b)式就很清楚，由于两个负数的和不可能是零，所以它就没有根。

如果方程中有一个未知数在根式中，其解法如下列规则所述。

移项，值得一个根式(如果有两个或两个以上的很式，则是最简单的一个)成为该方程一边的唯一一项。

其次，把得出的方程两边都乘到和很式指数一样高的幂次。

将(方程的)两边的同类项合并，如果根式仍然存在，则重复前面两种运算，然后再解这个方程。

验算。把所求得的值代入原方程，然后通过开方求根，而不是通过将方程的两边乘到任何幂次，将所得到的数字方程约化为最简单的形式。

最后，将全部奇根去掉。

11. 虚 数

当我们在研究代数不断前进时，几乎每前进一步都要涉及到应用一种更复杂、更精炼的数的形式。在算术中，正整数和正分数足以满足一切需要。在代数的一开始就引入负整数和负分数。二次方程如 $x^2-2=0$ 的解法使我们必需应用无理数，并引导我们研究根式的运算方法和解根式方程。每一种新的数都呈现为我们要解的方程的根，因此，随着每一个数的引入，我们代数方法的力量和普遍性也就增大了。

到目前为止，我们一直避免使用或弃置不用负数的平方根，因为它是一个虚数。我们的确不能想像 $\sqrt{-2}$ 这个数能量度任何长度。正数是我们量度时所需要的一切数。同样，如果一个人只需要极其简单的一点数学，他可能会认为，既然用正整数进行计算已足够了，那么，分数和无理数就不必要了。

正如我们已定义了并运用了负数和无理数的加、减、乘、除那样，我们现在就将定义所谓虚数的这些运算。引入这些数使我们能够完全解出各种情况的二次方程。它们在应用数学的许多分支，特别是电学理论中，常常用到这些数。

方程 $x^2+1=0$ 或 $x^2=-1$ 表明 x 是一个数，其平方为 -1。通过将一个新的数 $\sqrt{-1}$ 定义为其平方为 -1 的一个数，我们可求得方程 $x^2+1=0$ 的一个根。

同样，$\sqrt{-5}$ 是一个其平方为 -5 的一个数。并且，一般地，$\sqrt{-n}$ 是一个其平方为 $-n$ 的数。显然，$\sqrt{-5}$ 意味着与 $\sqrt{5}$ 根不相同的东

西，同样，$\sqrt{-n}$ 意味着与 $\sqrt{n}$ 很不相同的东西。

正整数是单位 $+1$ 的所有的倍数，负数是单位 -1 的所有的倍数。同样，纯虚数是虚数单位 $\sqrt{-1}$ 的实倍数，如 $2\sqrt{-1}$，$5\sqrt{-1}$ 和 $6\sqrt{-1}$。

此外，$\sqrt{-4}=\sqrt{4\cdot(-1)}=2\sqrt{-1}$；$\sqrt{-a^2}=\sqrt{a^2(-1)}=a\sqrt{-1}$；$\sqrt{-5}=\sqrt{5}\cdot\sqrt{-1}$。

虚数单位 $\sqrt{-1}$ 经常用 i 这个字母来表示；即 $3\sqrt{-1}=3i$。

假如把一个实数用加号或减号与一个纯虚数结合起来，这样得来的式子叫做复数。

12．对　数

人们发明对数来缩短涉及一个或多个乘、除、乘方和开方运算的冗长的数值计算的工作。对数的运用已把计算的劳动减轻到这样一种程度，以致原来不使用对数时一般需要好几小时的计算，现在由于有了对数的帮助，只需用原来时间的一小部分时间就行了。

如果我们写方程

$$n=b^a, \tag{1}$$

这里，我们表示了数 n 和以 b 为已知底的对数 a 之间的根本关系。用对数的记号，这可写成

$$\log_b n=a, \tag{2}$$

读作"以 b 为底时，n 的对数等于 a"。我们可以把对数和底用一句话定义如下：

一个已知数的对数是另外一个称为底的数为了等于该已知数而必须乘到的幂次。

重要的是要认识到方程(1)和(2)不过是表示完全相同的关系的两种不同的方法，一种是指数的方法，另一种是对数的方法。最重要的是必须记住，对数就是指数。

因此，在 $81=3^4$ 中，给定的数为 81，底是 3，对数是 4；即 $\log_3 81=4$。

常用对数的底为 10。因此，常用对数表实际上是数 10 的指数表。因为这些指数的大部分为无理数的近似值，所以，凡用对数来进行的计算，只能求得近似的结果。然而，有些对数表上的对数竟精确到 20 多位小数；因此，使用适当的对数表可以达到几乎任何需要的精确度。在数值计算中，几乎只使用常用对数。

在计算中使用的唯一的另一种对数称为自然对数。它的底为无理数 2.7182^{+}，通常用 e 这个字母来表示，并且主要用于理论的目的。

可以证明，上述支配有理指数用途的规律同样适用于无理指数。在用对数来进行（计算）工作时，这种事实必须承认。

13. 几何和几何术语

几何是数学的一个分支。数学的这个分支虽然用到数字，但主要不是研究数字。虽然它也使用方程，但它主要不研究方程。它主要研究图形，如三角形，平行四边形和圆。

立体有长度、宽度和厚度。面有长度、宽度，但没有厚度。线有长度，但既没有宽度，也没有厚度（线有直线和曲线）。点有位置，但没有大小。

射线是一根从某一点开始，无限止地延伸的线。从同一点出发的两条射线形成一只角。因此，两根射线就是一只角的两臂或两边。如果一只角的两边在一条直线上向两个相反方向延伸，则此角称为平角。平角的一半称为直角。锐角是比直角小的角。钝角是比直角大的角。优角是比平角大的角。

三边的直线图形称为三角形。等腰三角形是其中有两边相等的三角形。等边三角形是三边都相等的三角形。一只角为直角的三角形称为直角三角形。一只角为钝角的三角形称为钝角三角形。所有角均为锐角的三角形称为锐角三角形。三只角相等的三角形称为等角三角形。直角三角形中对直角的边称为斜边。

在一平面上，其所有点都与该平面上一固定点等距的封闭曲线称为圆。此固定点称做圆心。通过圆心，两端终止在圆上的直线叫做直

径。直径的一半为半径。(此)封闭曲线的长度称为圆周。

14. 三角函数和直角三角形的解

三角形的边和角是相互有关的。我们从几何学中可知道这点。三角学一开始就说明三角形的边和角间的关系的精确的性质。为此目的，三角学应用了边与边之间的比值。这些比值称做三角函数。在一三角形中，任一锐角 A 的六个三角函数表示如下:

$\sin A$, 读作"sine of A"

$\cos A$, 读作"cosine of A"

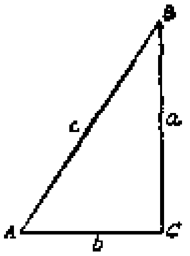
$\tan A$, 读作"tangent of A"

$\csc A$, 读作"cosecant of A"

$\sec A$, 读作"secant of A"

$\cot A$, 读作"cotangent of A"。

这些三角函数(比值)定义如下(见图)



(1) $\sin A=\frac{对边}{斜边}\left(=\frac{a}{c}\right)$;

(2) $\cos A=\frac{邻边}{斜边}\left(=\frac{b}{c}\right)$;

(3) $\tan A=\frac{对边}{邻边}\left(=\frac{a}{b}\right)$;

(4) $\csc A=\frac{斜边}{对边}\left(=\frac{c}{a}\right)$;

(5) $\sec A=\frac{斜边}{邻边}\left(=\frac{c}{b}\right)$;

(6) $\cot A=\frac{邻边}{对边}\left(=\frac{b}{a}\right)$。

这些函数(比值)在研究三角学中具有基本重要意义，必须记住。

三角的最重要的应用之一是解三角形。现在我们来解直角三角形。一个三角形由六个部分组成，三条边和三只角。解一个三角形就是要求出未知的部分。如果三角形的三个部分(其中至少有一个为边)为已知，

则此三角形就可解出。直角三角形的一只角，即直角，总是已知的。因此，如果它的两边，或一边和一锐角为已知，则此直角三角形可解。

解直角三角形的一般方法如下：

(1) 尽量正确地画一图形表示所论的三角形。

(2) 如一锐角为已知，则把它从 90° 中减去，以得到另一锐角。

(3) 为了求出未知部分，从(1)到(6)三角函数(比值)中选出一包含该未知部分和二已知部分的公式，并解公式求得未知部分。

(4) 检验求得的值。如果它们满足不同于在最后一步中所使用的那些关系的话，它们就是正确的。下列的关系是一种方便的数值检验：

$$a^2=c^2-b^2=(c+b)(c-b)$$

15. 三角函数的图形表示

变量是一个可以有无限数值的量。变量通常用字母表中最后几个字母如 x, y, z 来表示。

其值不变的量称为常量。数值或绝对常量在所有的问题中保持相同的数值，如2，5，$\sqrt{7}$，π，等等。任意常量是其值在某一特殊问题中固定的量。这种量通常用字母表中前面的几个字母如 a, b, c 等等来表示。

变量的函数是其值依赖于变量的值的一种量。几乎所有科学问题都要涉及这种量和这种类型的关系。另外，我们在日常生活的经验中也不断地遇到描绘一个量对另一个量的依赖关系的情况。如此，在其他条件相同的情况下，一个人所能举起的重量取决于他力量的大小。因此，我们可以把所举起的重量看作是人的力气的函数。同样，一个小孩能跑过的距离可以看作时间的函数。一个正方形的面积是其一边的长度的函数，一个球体的体积是其直径的函数。同样，三项式

$$x^2-7x-6$$

是 x 的函数，因为其值将取决于我们为 x 所假定的值，而

$$\sin A, \qquad \cos 2A, \qquad \tan\frac{A}{2}$$

都是A的函数。

变量所取的值与取决于此变量的函数的相应值之间的关系可由一个几何图象来清楚地加以表明。在图象中，以变量所取的值作为平面上一点的横座标，函数的相应值作为此点的纵座标。依次过这些点而画出的平滑曲线叫做函数的图形。下面是画一个函数图形的一般方法。

第一步。使y等于此函数。

第二步。让变量($=x$)取一些不同的值，并计算函数($=y$)的相应值，然后用表格的形式将此结果写出。

第三步。作出以x的值为横座标，以y的相应值为纵座标的点。

第四步。画一条平滑曲线，依次通过这些点，这条平滑曲线称为函数的图象。

16. 数和数字

首先，让我们将注意力转向分数。你(们)肯定遇到过像“一半”和“四分之一”这样的表达方式。当分数的分母等于2或4时，就用到它们。$\frac{1}{3}$读作“三分之一”。其他分数都是用同样的方法来读的。于是，我们将$\frac{1}{5}$，$\frac{1}{6}$，$\frac{1}{7}$，$\frac{1}{10}$，$\frac{1}{25}$，$\frac{1}{100}$读作五分之一，六分之一，七分之一，十分之一，二十五分之一，百分之一。这些表达方式被看作名词，因此，可以有复数。因此，我们将$\frac{2}{3}$读作三分之二(两个三分之一)；同样，将$\frac{5}{6}$，$\frac{9}{10}$，$\frac{5}{100}$读作六分之五，十分之九，百分之五。然而，如果分母的最后一个数字是一或二，那么，我们就不按上述方法来读这个分数。例如，我们将$\frac{5}{21}$读作“二十一分之五”。这种方法也用于其他的情况。如果分数不是一个普通的分数$\left(例如\frac{1}{1089}或\frac{501}{1205}\right)$，那么，我们就说“一千零八十九分之一”或“一千二百零五分之五百零一”。

其次，让我们来察看一下小数。它们读起来很简单。你(们)只要用普通的方法读(该数的)整数部分，然后说“点”(代表“小数点”)，然后

再一位一位地读小数。因此，12.65读作十二点六五。π准确到（小数点后）六位数处，等于三点一四一五九二。准确到五位有效数字，它等于三点一四一六。当小数小于一时，在英国通常不写0，例如不写0.56，而只写.56。.56读作“点五六”。.0007读作“点零零零七”或更普通一些，（读作）“点三个零七”。

现在谈到代数（表达）式。分数又读“分之”。$\frac{2a-1}{ax+b}$读作$ax+b$分之$2a-1$。括弧则用“乘”这个词来表示。例如，$(a+b)(a-b)$读作“a加b乘上a减b”。幂（次）则用指数或幂数来表示。指数2读作“平方”，指数3读作“立方”，或“三方”。其他指数（分别）读作四方，五方，负二方，n方”。恒等式

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

读作“a三方加上b三方等于a加b乘a平方减ab加b平方”。或方程

$$x^{-\frac{2}{3}}+\sqrt[5]{a^2}=0$$

读作“x负三分之二方加a平方的五次根等于零”。

有些整数只能被它们自身或一除尽，它们称为素数。有些整数不能用两个素数的分数来表示，因此，它们称为无理数。

17．分数（I）

如果a, b为两整数，由a和b相结合便形成一个新的数(a, b)，用通常的记法就是$\frac{a}{b}$，这个新的数规定能满足以下的条件：

（1）(a, b)被认为可以顺序地大于、等于或小于1。这里的大于、等于或小于的意思不是它们的原始意义指数量的大小，而是我们在整数的情况下所用的意义，指给这些数指定相对的阶。

（2）$(a, 1)$定义为等于a；因此，如果$b=1$，这种结合被看成是等于整数a。把（1）与此公设连起来，新数字的阶不仅它们之间对阶，而且连它们和整数之间相对的阶也都指定了，这样，整个集和分数阶都定了。这就是说，在两已知的数中总是可以说出那一个数的秩较高。

（3）两个分数的加法定义为：

$$(a, b)+(c, d)=(ad+bc, bd)$$

(4) 分数的乘法定义为:

$$(a, b)\times(c, d)=(ac, bd)$$

(5) 用分数作指数可以用以下的公设来规定:

$$x^{(a, b)}\times x^{(c, d)}=x^{(a, b)+(c, d)}$$

其中 x 是任一数，或者是整数，或者是分数。在这种解释成立时，符号 $x^{(a, b)}$ 要按照这个公设来解释。

我们将观察到，在 $b=1, d=1$ 的情况下，上述定义和在整数的情况下所采用的那些定义相一致；因此，新数和整数一起形成一个具有一致运算规律的集。很容易就知道用新数运算能满足交换律、结合律和分配律。除法的逆运算现在在数字的范围内总是可能的，因此

$$(a, b)\div(c, d)=(ad, bc)。$$

减法的逆运算 $(a,b)-(c,d)=(ad-bc, bd)$ 仅在 $(a, b)>(c,d)$时才是可能的。

一对整数的结合是一个“数”，和我们迄今为止所讨论的基数、序数都是数，大不相同。将“数”这个名词扩大到分数的正确性在于以下这个事实：对它们能够作出一套相容的运算方案，而其规律与只包括整数的运算的规律相一致。

18. 分数(II)

上面指出的那种方案足以说明分数性质的正式定义和逻辑发展，但是，它会受到反对，说它是任意性的；的确，不容易知道(看到)，除了由非算术的传统分数概念曾经提出以外，人们为什么会设想出这些特殊的运算规律。

为了弥补这种缺点，这里将对分数的本质提出一种看法，这种看法将分数和计数的过程紧密连系起来，说分数和整数与计算过程的关系十分相似。上面谈到的组合规律好像很自然地由于对分数的这种看法而来，除了第(5)条以外，这一条无论如何，它们是直接依据整数指数的规则提出来的。

考虑一个由 b 个物体所组成的集，从这 b 个物体中挑出其中任何 $a(\leq b)$ 个。如果我们将这 a 个物体不仅看成为数 a 的单一物体，而且看成是属于一个集，这个集的数 a 和它们所属的那个集的基数 b 是相伴的。如果这种过程与所用的特殊的集无关，那么，(a, b) 这个抽象分数与上述过程连系的方式就和数 b 与计算一个基数为 b 的集的过程连系的方法相类似了。因此，分数 (a, b)，或 $\frac{a}{b}$ 是每一个物体都属于 b 个物体的集中的 a 个物体的集的特征。当我们观察到所取的 a 个物体不一定非全部属于同一个 b 个物体的集不可时，显然就可以把定义扩大到 $a>b$ 的情况；只要把它们中的每一个看成为主要是属于基数 b 的某一个集就行了。按照这个观点，分数，譬如说 $\frac{3}{5}$，每种都属于五种东西(事物)的一个集中的任意三件东西(事物)的一个特征，即 $\frac{3}{5}$ 意指五中取出三。从五件东西(事物)中取出的三件必然在大小或某种别的数量方面应当完全相等，这一点和分数的真正本质毫不相干，这和五件东西(事物)必须是指五件相等的东西(事物)这个假设与数五的真正本质毫没关系是一样的。

19. 极限法

极限法在纯(理论)分析学以及应用分析学于几何和动力学上都是必不可少的，它在几何学上的来源是穷举法，这是希腊几何学家们在简单情况下用来决定长度、面积和体积的方法。这种方法经过数值无穷大这一概念的补充，后来以各种不同形式发展成为一种一般的方法，从而形成微积分学的基础。传统的几何学上的极限概念可用下面的实例来说明，即将一曲线的长度定为一系列经过选择的内接多边形的极限。当多边形的边数不断增多，而多边形各边的最大长度无限减少时，多边形的周长被认为不断趋近于曲线的预期长度。在被描述为使多边形的边数无限增多的过程结束时，极限，即曲线的长度，就被认为是真正达到了；然而，这种获得极限的方法对于感官来说，是难以想像的，它把从一个具有隅角，并以直线段为界的几何图形变为一个没有隅角，并以

曲线为界的几何图形的实质(际)的变化掩盖起来了。从几何直观来看，大家认为极限是显然存在的，关于这一点没有人有所怀疑。至于曲线具某一面积的长度，大家也认为是无须证明的。第一个认为极限的存在需要证明的数学家是柯西，他证明了连续函数的积分的存在。极限的传统方法的逻辑基础是有缺陷的，这点在近代已得到经验的证明。一是展示了连续函数没有微商，二是发现了过去认为是用极限方法分析的普通结果还有许多其它例外情况，这些情况是一直在从事于研究分析基础的那些数学家发现的。

算术上的极限理论已总结为收敛的一般原理，这对一系列数极限的存在提供了确定的准则；近代分析中相当大的一部分致力于求得根据无理数理论改写的一般准则的特别形式；因为，如若没有无理数的算术理论，则证明收敛序列的极限的存在的一切尝试，都将注定要遭到不可避免的失败；而这原因很简单，就是，有理数的收敛序列的极限不一定在这种数的范围之内。用有理数的收敛序列来定义实数并不仅仅是假设有这样序列的极限的存在；而且还引入了扩大了的数的概念，就是说，它具有这样一种特性，即有序实数的格式应当形成一个一致的整体，而且它是这样的，即在实数范围内的每一个数的收敛序列必然在那个范围内有个极限。实数集的存在这个公设是可以成立的，只要能证明：可以给这个集的元素制定一个完整的定义和公设的方案，并且这样的一套方案并不导致矛盾。至于上述的长度、面积、体积等等方面的极限的存在，其程序的次序与传统的相反，因为极限的存在不再是从几何的直观推演出来的。例如，在决定长度时，并不假定它是独立地存在的，而是把它定义为代表一系列适当内接多边形周长的那一系列数的算术极限。如果这个序列是收敛的，而其极限又不依赖于严格地选择符合于适当限制条件的多边形，那么，这样求得的极限就确定曲线的长度。如果极限不存在，该曲线则被看成是没有长度的。

20. 极限和无穷大

能接连取不同数值的一个数叫变量。应当记住：虽然这个变量接

连所取的值按照某种规律通常是互相关连的，但是，这些值相互之间并不一定有任何明确的关系。如前所说，变量普通是用 x，y 或 z 等字母来表示的。

因此，在方程

$$ax^2+bx+c=0$$

中，变量是 x。在这种情况下，变量接连所取的值是按这个代数方程所表示的规律而相互关联的。

循环小数 .9999… 的数值是依赖于在右边加多少 9 到序列上的一个变数。这样加上的每一个 9 都使 V 增大，而能这样加上的 9 的个数是无限的。尽管如此，V 总是小于 1，虽然它越来越接近 1 这个值。这里，1 这个数称为变量 V 的极限。

假如一个变量 V 接连取一系列的值，它们不断地越来越趋近一固定数 L，以致 $V-L$ 的数值变得始终小于任何有限数，不管它多么小，那么，V 可说是趋近于极限 L。

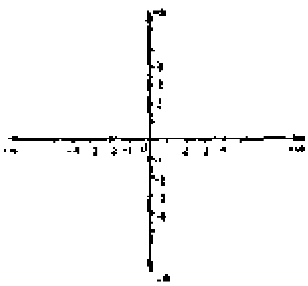
这可写成 V 的极限 $=L$。

符号 $\to$ 给我们 $V\to L$ 这等价的记法，读成 V 趋近 L 为其极限。

如果变量 n 接连取 1，2，3，4，…所有的值，我们可以想到 n 没有最后的值，因为自然数系是无限的。这里，我们可以说，n 无限地增大或 n 变成无穷大。这点可叙述如下：

定义。如果变量 n 变得始终大于任一正数 K，无论它怎样大，我们就说，n 无限地增大或 n 变成无穷大。

表示已变成无穷大的变量通常用 ∞ 这个符号，读作无穷大。



无穷大并不是像 3，$\sqrt{2}$ 和 -9 这样意义的数。它大于任何数。为了当前的目的，我们只能把它看作一种比喻，而不能看作一个能进行加、减、乘、除的数。事实上，我们不能按照基本运算的定律对 ∞ 这个符号进行运算。

我们可以有一个负的无穷大，也可以有一个正的无穷大。为了表

示 x 和 y 在图解上所能取的数值的范围，座标轴常如下图那样注明。

一个常数，不管它有多大，总不能说成是无穷大。

如果变量 $\frac{1}{n}$ 中的 n 接连取数值 1,2,3,4,…，不能设想 $\frac{1}{n}$ 有最后的数值。但是，随着变量 n 无限地增大，$\frac{1}{n}$ 则变得越来越小，并且越来越趋近于零。却又永不真正变成零。

一般说来，如果在分数 $\frac{a}{n}$ 中，a 是一个不是零的常数，而 n 是一个无限地增大的变量，则 $\frac{a}{n}$ 趋近于零为其极限。

21．平面解析几何中的基本概念

序　言

解析几何的发展通常归功于两个法国数学家费马(1601—1665) 和笛卡儿 (1596—1650)。解析几何的主要特性是它在研究几何时，使用了代数。还有一个虽然不大强调，但却仍很重要的特性是它允许把代数的关系用几何来加以形像化。在历史上，解析几何的发展为微积分学的发展，打开了方便之门。许多人都把微积分学看作开始于十七世纪的科学革命的基本数学工具。

座　标

解析几何的中心思想是在一平面上的点与一对数(x,y)之间建立一个一对一的对应关系。选择两条互相垂直的线和一个单位长度。

按照传统，我们称横轴为 x 轴，纵轴为 y 轴，它们的交点为原点。习惯上还将 x 轴的正向看作向右，y 轴的正向看作向上。每根轴均用选定的单位长度来标度，从而使 O 这个字母与原点相联系，使 $+a$ 这个符号与 x 轴上原点的右边 a 个单位的距离相连系，或者是 y 轴上原点上面的 a 个单位相连系。同样，$-a$ 与 x 轴上原点左边 a 个单位的距离相连系，或者 y 轴上原点下面的 a 个单位相连系。就这样，所有实数和 x

轴上的点之间以及所有实数与 y 轴上的点之间，建立了一种一对一的对应关系。

我们将根据以下规则把一对数字和平面上的任意一点相连系：我们从那一点各作一垂线到两根轴的每根轴上。如果这两根线在 a 处与 x 轴相交，在 b 处与 y 轴相交，我们将 (a, b) 这对数字指定给这一点。我们称 (a, b) 为 p 点的座标，我们写 $p(a,b)$ 来表示这个事实。x 轴上的 a 这个符号称为 p 点的 x 座标，或横座标；y 轴上的 b 这个数称为 p 点的 y 座标，或纵座标。反过来，如果我们打算寻找和 $p(a,b)$ 对应的点，我们就通过 x 轴上标了符号 a 的点作一条与 y 轴平行的线。再通过在 y 轴上标了符号 b 的点作一条与 x 轴平行的线。这两条平行线的交点用 $p(a, b)$ 来标记。

这两根轴将平面分成为四个象限，分别称为第一、第二、第三和第四象限，并用Ⅰ，Ⅱ，Ⅲ和Ⅳ来标记。第一象限中的各点的座标都是正数；在第二象限中，x 座标是负的，y 座标是正的；在第三象限中，座标均为负的；在第四象限中，x 座标是正的，y 座标是负的。

平面内的点集

我们已经表明了在一平面内的点与一对数字 (x, y) 之间的一种一对一的对应关系。平面内的某些点集可能特别有趣。例如，我们也许希望考察组成某圆周的点集，或者构成某三角形内部的点集。人们也许想知道这类点集是否可以用一种简洁的数学记号来简要地描述。

我们可以用写出

$$\left\{(x,y) \mid y=2x\right\} \tag{1}$$

来描述有序对 (x, y) 或者对应点的集，其纵座标等于横座标的两倍。于是，实际上，(1)中的垂线读作“其”，而所谓“有序对的集的图象”指的是平面上和有序对的集相对应的一切点的集。学生将会很容易地推断出构成图象的点位于一条直线上。

考虑一下集

$$\left\{(x, y) \mid y=x^2\right\}$$

和我们以前的解释相一致，这个符号代表有序对(x, y)的集，其纵座标等于横座标的平方。这里，整个图象构成一幅简单可识的几何图形，一条称为抛物线的曲线。

在这两个例子的基础上，人们也许会相信任一随意画的一根曲线，当然，它决定点的集或有序对的集，都能用一简单的方程来简明扼要地描述。遗憾的是情况并非如此。

现在考虑一下集(合)

$$\left\{(x, y) \mid y>2x\right\} \tag{2}$$

来描述(x, y)的点集，其纵座标大于其横座标的两倍。在这种情况下，我们的点集构成的不是一条曲线，而是座标平面的一个区域。

22. 锥　线

序　言

锥线的定义是一平面与一圆锥的交线。为了方便起见，我们将用成对正圆锥。

如果相交的平面平行于锥体(圆锥)的底，则相交曲线为圆。

如果相交的平面截一对锥体(圆锥)中的一个，而不与另一个相交，并且，如果平面不平行于底，则曲线称为椭圆。

如果相交的平面平行于圆锥的一元素，相交的曲线称为抛物线。

如果相交的平面是与两个圆锥都相交，则相交的一对曲线称为双曲线。

我们让学生们自己来证实平面与圆锥交线在某些蜕化的情况下，也许是一点、一条线或两条线。

这些曲线早在公元前350年希腊数学家米尼克穆斯就研究过。但是，要到一百多年以后，才有沛加的阿波罗纽斯用严密而艰苦的欧几里德几何方法对这些曲线进行透彻而详细的分析。阿波罗纽斯关于圆锥的研究为他赢得了“伟大的几何学家”的称号。按照现代的标准，阿波罗纽斯所用的方法是非常讨厌的。由费马和笛卡儿所发展的座标几何

把锥线曲线这个难题简化到中等才能的学生所易于接受的程度。

阿波罗纽斯所推演出来的圆锥曲线的性质代表了他的特殊的理解力和智慧。但是，座标几何在阿波罗纽斯所得到的性质以外，还可以推演出圆锥曲线的无穷性质。阿波罗纽斯所忽略的许多性质在应用上变得远远比他所推演出来的重要得多。

人们之所以挑出这些曲线来加以特别研究，不仅因为它们具有历史上的重要性，而且因为有许多物理现象可以用这些曲线来很好地叙述。例如，一行星在它围绕太阳公转时所遵循的轨道是个椭圆。一抛射体只受重力影响时所循的轨道是抛物线。一些彗星的轨道是双曲线。在我们的一些问题集中，将会遇到有锥线的物理现象。

既然每条锥线都具有无限多的性质，而任一性质都足以把它唯一地定义出来，我们将选择一种与它实际用途密切相关的性质来定义每一曲线的性质。

23. 矩阵的应用

近年来，在数学和许多各种不同的领域中，矩阵的应用一直以惊人的速度不断增加。在研究量子力学时，矩阵理论在现代物理学上起着主要的作用。解决应用微分方程，特别是在空气动力学，应力和结构分析领域中的问题，要用矩阵(方)法。心理学研究上的一种最强有力的数学方法是因子分析，这也广泛使用矩阵(方)法。近年来，在数学经济学和商业管理问题方面的发展已经导致广泛地使用矩阵(方)法。生物科学，特别在发生学方面，用矩阵(的)技术很有成效。不管学生主要兴趣是什么，矩阵基本原理的知识可能扩大他能读懂的文献的范围。

在本节中，我们将列举一些如何利用矩阵的初步例子。

解一有 n 个未知数的 n 个联立一次(线性)方程是应用数学的一个重要问题。解析几何的发明者和现代代数记数法的创始人之一笛卡儿相信，所有的问题最后都能约简为解一组联立一次方程。虽然这种信念现在认为是站不住脚的，但是，我们知道，从许多不同的学科里的一大群重要的应用问题都可以约化为这类的方程。许多应用要求解大量

的，往往数以百计的联立一次（线性）方程。计算机的发明已经使得矩阵(方)法在解这些难以解决的问题方面非常有效。

例题 1. 解联立方程求 x_1, x_2 和 x_3

$$2x_1+3x_2+4x_3=4$$
$$2x_1+x_2+x_3=-2$$
$$-x_1+x_2+2x_3=2$$

解：我们可以将这些方程写成矩阵的形式

$$\begin{pmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} \tag{1}$$

并将系数的矩阵称为 A，未知数的 3×1 矩阵称为 x，右边的 3×1 矩阵称为 k。于是，我们可以将方程(1)写成下式：

$$Ax=k$$

如果有可能求得一个 3×3 的矩阵，它用 A^{-1} 来表示，并称为矩阵 A 的逆矩阵，使

$$A^{-1}A=I \tag{3}$$

其中 I 是单位〔矩〕阵，那么，我们就用 A^{-1} 来乘方程(2)的双方。于是，方程(2)就会变成

$$A^{-1}Ax=A^{-1}k \tag{4}$$

用方程(3)，我们就能将方程(4)写成

$$Ix=A^{-1}k$$
$$x=A^{-1}k \tag{5}$$

特别为了这种情况不告诉你怎样得到它的.

$$A^{-1}=\begin{pmatrix} -1 & 2 & -1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{pmatrix}$$

将此用于方程(5)，我们得到

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} -1 & 2 & -1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$=\begin{pmatrix}(-1)(4)+ & (2)(-2)+(-1)(2)\\ (5)(4)+(-8)(-2)+(-6)(2)\\ (-3)(4)+ & (5)(-2)+ & (4)(2)\end{pmatrix}$$

$$=\begin{pmatrix}-10\\ 24\\ -14\end{pmatrix}$$

于是，$x_1=-10, x_2=24, x_3=-14$。从上面的讨论，我们看到解有 n 个未知数的 n 个联立一次（线性）方程的问题化成求系数的矩阵的逆矩阵的问题。因此，在矩阵论的书中，用大量的篇幅来讲求逆矩阵的技巧就不奇怪了。当然，我们在这有限的叙述中不会讨论这类的技巧。矩阵(方)法不仅在解联立方程中有用，而且在发现方程组是否相容，即方程组是否有解的问题，以及方程组是否是确定的，即是否只有一解等方面，都是有用的。

矩阵应用的另一领域是变换论。下面就是旋转变换的方程。

$$x=x\cos\theta+y\sin\theta$$

$$y=-x\sin\theta+y\cos\theta$$

这些方程可写成矩阵形式如：

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

一般地，我们(可)将任一 2×2 矩阵譬如说

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

看成依照矩阵方程把一点 (x, y) 变换成为一点 (X, Y)

$$\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \tag{6}$$

一种复杂的变换常常可以看作一系列简单的变换。在这类情况下，可能用一单一的矩阵来代表这个一系列的简单变换。考虑由列向量 $\begin{pmatrix}x\\ y\end{pmatrix}$ 与 2×2 矩阵，A_1，A_2，和 A_3 连乘所完成的一系列变换。这些连乘将产生一个我们可用列向量 $\begin{pmatrix}x\\ y\end{pmatrix}$ 来表示的点。于是，我们用

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A_3\left\{A_2\left[A_1\begin{pmatrix} x \\ y \end{pmatrix}\right]\right\} \tag{7}$$

来表示变换。

由于矩阵乘法是结合的，我们可以将方程(7)重写成

$$\begin{pmatrix} X \\ Y \end{pmatrix} = (A_3A_2A_1)\begin{pmatrix} x \\ y \end{pmatrix}$$

或

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

其中 A 是单个的矩阵，即矩阵 A_3, A_2, A_1 的积。

例题 2 对列向量 $\begin{pmatrix} x \\ y \end{pmatrix}$ 进行一系列与矩阵

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \text{和} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

对应的变换。用单一的一个变换矩阵来表示这一系列变换，并说明每一变换的图象意义。

解:

$$\begin{aligned}
\begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} (-1)(1)+(0)(0) & (-1)(0)+(0)(-1) \\ (0)(1)+(1)(0) & (0)(0)+(1)(-1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

这样，这一系列的变换将 (x, y) 变回到它自身。

24. 线性系统的图解

有两个未知数之线性方程或有两个未知数的线性系统的作图，取决于几个假设和定义。这些是：

I. 有两根互成直角的直线，$X'OX$，称为 x 轴，$Y'OY$ 称为 y 轴。

II. 在这些(直)线上有一个距离的单位。

III. 从 y 轴到纸上任何点的距离(平行于 x 轴来量度)为此点的 x 距离(或横座标)，从 x 轴到点上的距离(平行于 y 轴来量度)为此点的 y 距离(或纵座标)。

IV. 一点在 y 轴右边的 x 距离由一正数来表示，一点在 y 轴左边的 x 距离由一负数来表示。同样，一点在 x 轴上方的 y 距离由一正数来表示，在 x 轴下面的 y 距离由一负数来表示。简而言之，从轴向右和向上测得的距离是正的；从轴向左和向下测得的距离是负的。

V. 纸面上的每一点都对应于一对数，其中的一个或两个数可能是正的、负的、整数或分数。

VI. 一对已知数中，第一个是 x 距离的量度，第二个是 y 距离的量度。

座标轴的交点称为原点。

x 距离和 y 距离的数值经常称为点的座标。

方程与其图象之间的关系可叙述如下：

直线的方程是此直线上的任意点的 x 距离和 y 距离的数值都能满足的。

如一任意点的 x 距离和 y 距离的值满足此方程，则此任意点在此方程的图象上。

两个未知数的线性方程的图象是一条直线。因此，在作这样一个方程的图象时，只须找出两个其座标能满足此方程的点，然后，通过这两点画一直线。最方便的方法通常是找出这条直线与两轴相交的两点。然而，如果这两点靠得很近，就不能准确地定出直线的方向。只要选用相隔较远的两点就可以避免这种误差。

含有两个未知数的线性系统的图象解法是在相同的两轴上以同样尺度画出两个方程，并且从这一图象求得这两条直线交点的 x 值和 y 值。

通过方程的图象研究，我们把几何和代数这两门看起来似乎是完全不同的学科结合起来，并且学会用一种学科的语言来解释另一种学科的问题。

25. 作为科学语言的数学

科学的一个分科研究一类事物，该类的各成员中的变化，以及这些成员之间的关系。因此，自然科学的理想形式是和数学的理想形式一样的。自然科学的目的是去发现这样一些关系，它们能断定在某种情况下有事件 P 出现时，一定也有事件 Q 出现。当一门科学从描述和定性的阶段进展到可以用定量和解释的方式来表达关系的阶段时，该科学就采取数学的形式。天文学曾一度是一种描述性的科学，但是，开普勒和牛顿的工作建立了可以用数学来表示天体运动规律的基础。正是在这种意义上，数学有时称为科学的语言。当一门科学的公设满足一门数学的公设的要求时，那么，那门科学的假言命题以及从它们得出的推论就可以用来证实那门相应的科学的命题的预言。

一种科学理论和与它互相关连的数学体系之间的差别之一就是，如果数学的推论预言了和实验相矛盾的现象，那么，(该)科学的理论的一些或全部初始公设必须修改或抛弃；但是，虽然自然科学的（物理学的)理论已经失败，但是数学的体系并不受到怀疑，仍然和过去一样地始终一贯。它在阐明这些科学假设的不足方面，已经达到了它的一个目的。

有时，数学的发展已远远超过任何一门具体科学的需要，希腊数学家阿波罗纽斯所发现的圆锥曲线的性质，一直到开普勒利用椭圆形来描述行星绕着太阳运动的时候，才得到了应用。另一方面，科学上的发现有时进步得很快，以致适当的数学体系落后了。原来从表面上十分任意的几套公设出发作为抽象科学发展出来的重要数学理论，后来在数

学的应用上往往成为有用的工具。正是根据对代数方程的研究，数学家们才能预知在矿物学中只能找到32种晶型。光的圆锥折射，是首先由哈密顿从数学的研究中预见到，然后才在他的实验室中观察出来的。

数学的推理启发实验，同时，为科学目的而发展的数学工具也曾转而成为理论数学成长的强大刺激。对热在金属板中的流动的研究使物理学家傅里叶发明了级数，这不仅解决了热学研究上的复杂问题，而且也大大推动了理论数学的发展。

科学和数学齐头并进，相互帮助，相互促进。当科学的概括采取定量的形式，并经常启发新的实验时，就要数学的假言命题来起作用了；另一方面，科学上观察到的数据非常复杂，这就刺激了数学的发展，也扩大了它的基础。

26. 函数、变数和常数的概念

I

在数学中，很少有一个概念象函数的概念那样，起那么重要的作用的。因此，需要知道这个概念是怎样发展起来的。

这个概念像许多其他概念一样，起源于物理学。物理的量是数学的变量的先驱，它们之间的关系在16世纪后期称为函数关系。

例如，代表一物体在若干秒 t 中下落的若干英尺 s 的公式 $s=16t^2$ 就是 s 和 t 之间的函数关系。它描述 s 随 t 变化的方式。对这种关系的研究导致了18世纪的人们认为函数关系只不过是一个公式罢了。

只有在19世纪初期现代分析出现以后，函数的概念才得以扩大。在扩大了的意义上讲，函数可定义如下：如果一变量 y 随另一变量 x 而变化，即 x 的每个值都和 y 的一定值相对应，那么，y 就是 x 的函数。这个定义甚至在今天还适用于许多实际用途。

至于如何建立这种对应的关系，这个定义并未详细规定。可以如18世纪的数学所假定的那样，用公式来建立，但同样也可以用统计表那样的表格或用某种其他的描述方式来建立。

典型的例子是室温，这显然是时间的一个函数。但是，这个函数不能用公式来代表，但可以用表格的形式来记录或者用一种自动装置以图表形式来追踪。

现代给 x 的一个函数 y 所下的定义只是从一个空间 X 到另一空间 Y 的映射。当 X 空间的每一点 x 有一个确定的像点 y，即 Y 空间的一点，那么，映射就确定了。这个映射概念接近于直观，因此，很可以作为函数概念的一个基础。此外，由于这个现代的定义中体现了空间概念，所以，它的普遍性对函数概念的普遍性有很大贡献。

II

像时间、长度、面积、体积、质量、速度、压力、温度等等物理量的数值都是由量度来测定。而数学研究的量都不含任何特（殊）定的内容。从现在起，当谈到量时，我们考虑的是它们的数值。在各种不同现象中，某些量的数值是变化的，而另一些量的数值则保持不变。例如，在一点的匀速运动中，时间和距离不断变化，而速度则保持不变。

变量是连续取各种不同数值的量。常量是数值保持不变的量。我们将用字母 $X, Y, Z, \cdots$ 等等来代表变量，用字母 $a, b, c, \cdots$ 等等来代表常量。

在数学上，常量常常被看作变量的一种特殊情况、其数值都是一样的。

在考虑特定物理现象时，可能发生这种情况，即同一个量在某一现象中是常量，而在另一现象中却是变量。例如，匀速运动的速度是常量，而在等加速运动的速度则是变量。在各种情况下都具有同一数值的量称为绝对常量。例如，圆周与直径之比是绝对常量：

$$\pi=3.14159。$$

27. 函　数

连续变量。微分学里所研究的量，一般认为在大小上是可以连续变化的。这种变量可以用沿着一条直线运动的某一点和在线上所取的

某一固定原点之间的距离（例如x）来代表。于是，x的变化率就由动点的速度来代表。而此变化的连续性就意味着x不是突然地，而是逐渐地从一个值转到另一个值，因而能在某时能取其间的每一个x的值。

如果这个运动是匀速的，于是，在相等的时间内就通过相等的空间，那么，速度是常量，因而x的变率也说成是常量。它的量度就是在一个单位时间里所通过的空间，或是所得到的增量；当x增大时，就认为它是正的，当x减小时，就认为它是负的。

无论考察什么，至少会出现两个这样的变量；并且，如果只有两个变量，那么，其中一个变量的值一定要取决于另一个变量的值。后者称为自变量，通常用x来表示。用y来表示另一个变量，或者叫因变量时，就说它是x的函数，而把它对x的依赖关系用一般的方程

$$y=f(x)$$

来表示。

当这种关系用一定的符号如$y=x^2$，$y=\sin x$，$y=\log x$来表示时，那么，y就是x的一个已知函数，而且我们可以象在解析几何中一样，用某点在一平面上的位置来表示x的任何值以及y的对应值。为此目的，我们使用直角座标，该点自变量x为横座标，函数y为纵座标。x值的不断变化，引起了y值的不断变化，而这种变化可用平面上的点(x, y)、或说点P的运动来表示。因此，动点P从有着一定的x值和对应的y值的一定点出发，在平面上描绘出一条直线成曲线。这条线称为函数的图象。

以已知函数$y=x^2$为例，当$x=0$，$y=0$时，那代表点是原点。从这个位置出发，x增大时，y也增大，起初是慢慢的，以后越来越快，此动点P描绘了抛物线在第一象限中的分支。另外，假如从这一支的任何点出发，x减小，那么，此动点沿着曲线折回，通过原点后，描绘出第二象限中的分支。因此，整个抛物线是函数$y=x^2$的图象。

在考虑点P在描绘函数图象的运动时，我们为了使概念固定，假定纵座标足R（我们称之为P点运动的水平分量）的运动是从左向右的匀速运动。换句话说，x的增加率是常量。这图象立即表示出作为纵座

标的函数，在运动中是随 x 的增加而增加，还是随 x 的增加而减小。在第一种情况下，它叫 x 的增函数，在第二种情况下，它叫减函数。当然，应当注意到 P 本身的斜向运动在此处没有考虑。但是，假如我们选择用 x 轴的平行线 PS 将 P 投影到 y 轴上，那么，S 的运动将表明我们所关心的 y 的增减。如果这个函数是增函数，当 R 向右移动时，S 将向上移动，P 将倾斜地向上移动。这时，我们说，这图象的梯度或坡度是正的。因此，正坡度意味着增函数，而负坡度意味着减函数。

一函数可能在 x 值的某一范围内是增函数，而在值的另一范围内是减函数。例如，x^2 的图象表明 x^2 对于 x 的正值是一个增函数，对于 x 的负值是减函数。通过画 $\log x$ 或其方程为 $y=\log x$ 曲线图象，同学们将会看到(知道)此曲线将渐渐地说明 $\log x$ 总是 x 的增函数这一事实。同时，曲线 $y=\sin x$ 表示 $\sin x$ 有时为增函数，有时为减函数。

28. 数系的发展 (I)

数系如何通过概括数的概念而发展的过程是数学上一种颇有教益的研究。数这个词的原意是整数，也称做自然数，以便与称做人为数的各种不同类型的数相区别。数系在范围扩大以后也包括人为数在内。

整数。 当人类能区别各类事物而无须加以描述，即区别五条鱼和五这个数时，便开始有了数系。即使最原始的部落也至少也有几个字表达一些数的概念。巴比伦人的楔形文字记录有平方，有立方，还有一些级数，这表示在远古时代，整数系统就有相当大的发展。

古代整数记数法的体系很笨拙，在某些情况下，计算是借助辅助的和记录的手段而进行的。后来就以算盘代替了。这种计算器具有一个框子，还有算珠在线上或沟内滑动，至今仍在中国、日本、苏联、印度和其他东方的国家里使用。

分数。 数的其次一种类型是由于重分财产的实际需要而产生的。像$\frac{1}{2}$，$\frac{1}{3}$，$\frac{1}{4}$，… 这类分子为 1 的分数，对于简单的再分已足够应用了。公元前 1400 年 Ahme 的纸草上描述了将 $\frac{3}{4}$ 和 $\frac{5}{7}$ 这类分数变换成以 1

为分子的分数的方法。如

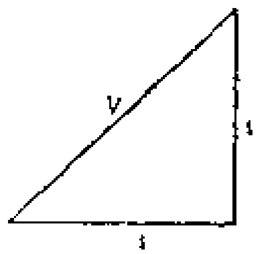
$$\frac{3}{4}=\frac{1}{2}+\frac{1}{4}, \qquad \frac{5}{7}=\frac{1}{2}+\frac{1}{7}+\frac{1}{14}$$

于是，$\frac{3}{4}$和$\frac{5}{7}$的加法能用将那些简单分数相加的办法来进行。在商业的实际问题中，涉及分数的计算用含有小单位的表来加以简化。我们钟表上计时的分和秒是巴比伦人的六十分数法的残余。

无理数。发展的第三阶段是发现有些线段不能用选定为单位长度的线段的整数或分数的倍数来表示。这种发现是与直角三角形的研究有关连的。毕达哥拉斯定律说：直角三角形两边平方的和等于斜边的平方。

希腊数学家知道一些可以用整数来表示它们这种关系的直角三角形，例如，三边之比为3,4,5和5,12,13的直角三角形。然而，如果直角三角形是等腰的话，他们认为斜边的长度与其它任一边是不能公度的。因为，如果两边的长度取作单位长度，那么，用(下面的)方程得出斜边：

$$V^2=1^2+1^2=2 \qquad (1)$$



V 没有整数值能满足该方程。如果有一分数$\frac{a}{b}$，其中 a 和 b 都是整数，那么

$$\frac{a^2}{b^2}=2 \text{ 或 } a^2=2b^2 \qquad (2)$$

我们假定 a 和 b 没有公因子，因为这种公因子一开头时就能约掉。从(2)得出 a^2 是个偶数，因此，a 必然是偶数，譬如说 $a=2a'$。将 a 的值代入(2)，得出 $4a'^2=2b^2$，因此

$$b^2=2a'^2$$

所以，b^2 是偶数，因此 b 也是偶数。于是得出 a 和 b 有因子2。但是，这点和 a 和 b 没有公因子的假设相矛盾。因此，可用分数$\frac{a}{b}$来表示斜边的这种假设导致矛盾，所以是不成立的。

这种“间接证明”表明符号$\sqrt{2}$不能和任一整数或分数对应，并且证明了有些数量是不能用这两类数字中的任何一种来表示的。

零。 下一次进展是约在公元600年发生在印度。他们发明了符号0来代表原来在算盘上计算时产生的空位。

引进符号零和发明位置原则，从而使同一数的符号可以按其所占位置的不同而用于不同的意义，大大改进了记数法，以致用符号计算比用算盘计算要快得多。整个中世纪，这两种计算方法的支持者一直不停地进行激烈的争论，其激烈的程度并不亚于和我们现代所进行的有关米制和英制单位孰优孰劣的争论。

29. 数系的发展(II)

负数。 数的概念发展的第五阶段也归功于印度数学家，并且是从研究代数方程而得到的。如下形式的方程

$$x+a=b, \quad x^2=a$$

按照其未知数是否代表一个包括在已被接受的数系中的数而分为是否可能的两类。

这样，$x+3=5$ 是一个可能的方程，因为它的根是整数 2，而 $x+5=3$ 则是一个不可能的方程，因为

$$x=-5+3=-2$$

不是一个已知的数。这类称为负数的数在人们观察到它们能用来代表债务时而最终为人们所接受。因此，在数的概念的这种扩展中，数量大小的概念上又加上了方向的概念。从负数的概念演化出称为向量分析的有向数量学说，已成为数学物理学上的有力工具。

有理数。 正负整数、分数和零的总体称为有理数类。

两个整数的和与积总是整数。这种性质用下面的话来表示，即整数类对加法和乘法的运算而言是封闭的。

对整数进行的逆运算减法和除法，并不总是产生整数。消除例外的愿望构成了有理数类，这类里所有的数都能借助于有理运算：加、乘、减、除，从一中求得到。有理数可借助一直线上的点，用图象来表示。任选两点代表零和一，用这两点之间的距离作为尺度来决定线上一点所代表的每一正的或负的有理数。

数轴上的间隔，不管多么小，里面总有有理数，这种性质在下面的陈述里表示了：有理数的集是处处稠密的。

实数。尽管有理数有稠密性，它们并不足以代表轴上的每一点。和无理数$\sqrt{2}=1.414\cdots$相对应的点并没有有理类的任一数来代表。可以证明：在数轴上的每一点都能指定一个小数，这小数可以是有限的，也可以是无限的；假如是无限的，既可以是循环的，也可以是非循环的。

有理数可以用有限小数或循环无尽小数来表示，例如：$\frac{1}{8}=0.125$，以及$\frac{1}{3}\cdot\frac{0}{3}\cdot\frac{4}{3}=0.312312312\cdots$，而无理数则是由不循环的无限小数来表示的。

轴上不与有理数相对应的任何一点P都将与无限非循环小数$a+0.a_1a_2a_3\cdots$相对应，这个数叫做对应于点P的实数。

这些无理数按它们是否是代数方程的根而分为代数的和超越的无理数两类。超越无理数的例子有圆周与直径的比率3.14159，以及自然对数系的底$e=2.718\cdots$。

$2^{\sqrt{2}}$这个数最近已经证明是超越的。有理数和无理数一起构成了实数类。

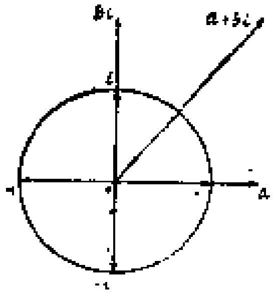
30．数系的发展（III）

复数。虽然无理数起源于几何学，但负数和复数都起源于代数学。

丢番图（公元300年）解二次方程的方法和我们现在解方程的方法大致一样，得出解的形式：$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$。起初，只有根式下面的数是一完全平方，才认为可能有解，并且只承认属于正号的根。

印度的巴斯卡拉（公元1114年）承认平方根的双重（正、负）符号用无理数计算，但是，认为负数的平方根是不可能的。

卡尔登于1545年发表了塔尔塔格里亚的三次方程的代数解，并用代入法和形式运算证明了$a+b\sqrt{-1}$这样一个复数满足方程，但是，他并不承认这种数，因为他没法解释它们。



阿干德于1806年提出了虚数的几何表示法。

为了使关系式$\sqrt{a}\cdot\sqrt{a}=a$，其中a是一正数，也可以适用于负数，他假设$\sqrt{-1}\cdot\sqrt{-1}=-1$。把这个关系式写成$\frac{-1}{\sqrt{-1}}=\frac{\sqrt{-1}}{-1}$的形式就表明$\sqrt{-1}$是1和$-1$之间的比例中项。利用决定两线段之间的一比例中项的作图方法，可以求得符号$\sqrt{-1}$的几何表示法。用实数轴的原点作圆心，以1为半径作一单位圆，并在0点上作一垂线和单位圆相交。阿干德用这根单位长度的垂直于代表1的线段的有向线段来代表$\sqrt{-1}$，它通常用i这个符号来表示。

像bi这样的虚数是用在原点上垂直于实数轴的直线上的点来表示的。

将复数与平面上的点相结合就取得进一步的发展。于是，复数$a+bi$就用笛卡儿座标为(a, b)的点P来表示。

复数$a+bi$的另一表示法是从原点O延伸到P点的有向线段。

有了这种几何解释，像$x^2=-1$及$x^2-x-1=0$这类方程就不再被认为是不可能的了，因为它们的根$\pm\sqrt{-1}$和$\frac{+1\pm\sqrt{-3}}{2}$好像是实数一样地容易表示了。

复数特别适用于表示二维的关系。它们原来是因为满足数学家们解释方程的所谓不可能的解的愿望而被接受的，但现在已经成为数学物理学上研究电、磁或热在金属板上的流动所必不可少的手段了。它们也用于制图的理论和流体力学上。虚数在"虚"这个涵意上来说，现已名存实亡。

复数代数的数系分类如下：

$$\text{复数}\begin{cases}\text{实　数}\begin{cases}\text{正　数}\begin{cases}\text{有理数}\begin{cases}\text{整　数}\\\text{分　数}\end{cases}\\\text{无理数}\begin{cases}\text{代数的}\\\text{超越的}\end{cases}\end{cases}\\\text{负　数}\end{cases}\\\text{虚　数}\end{cases}$$

复数系对于六个基本运算来说，可以称为封闭的，因为如果将这些运算中任一运算应用到复数上，决不会产生新型的数。

数概念的每一概括在运算的意义上都发生变化。符号＋和× 在复数的加法和乘法中的意义和它们在用整数、有理数或无理数进行运算时的意义大不相同。

两个复数的乘法。 $A\times B$ 的积可以几何方法求得，即按照使单位向量 OI 与 OA 相合所必须旋转的角度和必须延伸的比例来旋转向量 B 的角度和改变它的长度。这很容易办到，只要作三角形 OBP，使它和三角形 OIA 相似就行了。

在整个数和运算的概括中，加法和乘法的结合律、交换律和分配律始终不变。

数的概念的进一步发展是通过代数的基本公设中的变化来实现的。维尔斯特拉斯证明了：如果所有的公设都保持不变的话，那么，要构成比复数更一般的数类是不可能的。